Principles of Communications

Chapter II: Signals

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Classification of Signals $s(t)$

Deterministic Signals and Random Signals

- Deterministic signals: the values at any time are deterministic and predictable. $s(t) = \sin \omega t$.
- Characteristics: Frequency domain

$$s(t) \quad \xrightarrow{\text{Fourier}} \quad S(f) = \int_{-\infty}^{+\infty} s(t) e^{-j2\pi f t} dt$$

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- Random signals: the values at any time are random. $s(t) = \sin \omega t + \varphi$, $\varphi \in U(0, 2\pi)$.

Question: How about communication signals?, How to characterize random signal?

Characteristics of Random Variable

Random variable. 1-dim

- Distribution function: $F(x) = P(x \leq X)$.
- Probability density function: $f(x) = \frac{dF(x)}{dx}$.
- Frequently used random variables
  - Uniform variable: $f(x) = \begin{cases} 1/(b_1 - b_2), & b_1 \leq x \leq b_2, \\ 0, & \text{otherwise} \end{cases}$
  - Normal (Gaussian) variable:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp[-\frac{(x-a)^2}{2\sigma^2}],$$

    where a is the mathematical expectation and σ is standard deviation.
  - Rayleigh variable: $f(x) = \frac{2x}{a}\exp(-\frac{x^2}{a})$,
    where $a > 0$ is the mathematical expectation and $x \geq 0$.
- Question: How about n-dim variables?

Numerical Characteristics

- Mathematical Expectation:

$$a = E(x) = \int_{-\infty}^{\infty} x f(x) dx$$

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- Variance:

$$D(x) = E[(x - E(x))^2] = E(x^2) - a^2$$

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- Moment:

$$E[(x-a)^k] = \int_{-\infty}^{\infty} (x-a)^k f(x) dx.$$

  - If a=0, it is called k-th origin moment.
  - If $a = E(x)$, it is called k-th central moment.
- Question: Illustrate properties of the above numerical characteristics.

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Basic Concepts of Random Process

Everywhere in communication systems.

How to describe random process mathematically?

- n-dim pdf: $f(x_1, \cdots, x_n; t_1, \cdots, t_n)$.
- Mathematical expectation: $E(\xi(t))$
- Variance: $D(\xi(t)) = E[\xi^2(t)] - E^2(\xi(t))$.
- Auto-correlation function: $R(t_1, t_2) = E[\xi(t_1)\xi(t_2)]$.

Stationary Random Process

If the statistic characteristic of a random process is independent of the time origin, it is called $\sim$. (strict pdf – $f(\mathbf{x}, \mathbf{t}) = f(\mathbf{x}, \mathbf{t} + \boldsymbol{\tau})$) (generalized – numerical characteristic)

- Mathematical expectation: $E(\xi(t)) = \text{const}$.
- Variance: $D(\xi(t)) = E[\xi^2(t)] - E^2(\xi(t)) = \text{const}$.
- Auto-correlation function: $R(t_1, t_2) = R(\tau)]$, where $\tau = |t_1 - t_2|$.

Question: $\xi(t) = \sin \omega t + \varphi$, $\varphi \in U(0, 2\pi)$, generalized stationary?

Ergodicity

- Definition: If a random process has ergodicity, its statistic mean is equal to its time average. "Time average" of numerical characteristics, such as mathematical expectation and auto-correlation function, is equal to these of "statistic mean". (Sometimes, we just say stationary random process.)
- Mathematical expectation: $E(\xi(t)) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \xi(t)dt$.
- Autocorrelation function:
  $R(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \xi(t)\xi(t+\tau)dt$.
- Use the above two properties to demonstrate the ergodicity.
- Question: $\xi(t) = \sin \omega t + \varphi$, $\varphi \in U(0, 2\pi)$, ergodicity?

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Characteristics of $R_\xi(\tau)$ and PSD

Characteristics of $R_\xi(\tau)$

- $R_\xi(0) = E[\xi(t)^2]$ – power;
- $R_\xi(\tau) = R_\xi(-\tau)$; – even function;
- $|R_\xi(\tau)| \leq R_\xi(0)$;
- $R_\xi(\infty) = E^2(\xi(t))$ – DC power;
- $R_\xi(0) - R_\xi(\infty) = \sigma_\xi^2$.

Power spectral density: Wiener – Khinchin theorem

The autocorrelation function $R_\xi(\tau)$ and power spectral density $P_\xi(f)$ of a stationary random process are a pair of Fourier transform, i.e.,

$$P_\xi(f) = \int_{-\infty}^{\infty} R_\xi(\tau) e^{-j\omega\tau} d\tau. \quad R_\xi(\tau) = \int_{-\infty}^{\infty} P_\xi(f) e^{j\omega\tau} df.$$

Gaussian Process

Definition: n-dim Gaussian variable

- $$f_\xi(x_1,\cdots,x_n;t_1,\cdots,t_n) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}}\exp\left(\frac{1}{2}(\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu)\right.$$
  where $\mathbf{x} = [x_1,\cdots,x_n]^T$, $\Sigma = E[(\mathbf{x}-\mu)(\mathbf{x}-\mu)^T]$ is covariance matrix, and $|\Sigma|$ is the determinant operator.
- If x_i, $i = 1,\cdots,n$ are independent, we obtain

$$f_\xi(x_1,\cdots,x_n;t_1,\cdots,t_n) = \prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}\sigma_i}\exp\left(-\frac{(x-\mu_i^2)}{2\sigma_i^2}\right).$$

Special Function

- Error function: $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-z^2}dz$.
- Complementary error function:
  $\mathrm{erfc(x)} = 1-\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-z^2}dz$.

Gaussian White Noise

White Noise

- Auto-correlation function: $R(\tau) = \frac{n_0}{2}\delta(\tau)$.
- Power spectral density: $P(f) = \frac{n_0}{2}$.

Band Limited White Noise

- Power spectral density: $P(f) = \frac{n_0}{2}, \quad -f_H < f < f_H$.
- Auto-correlation function: $R(\tau) = n_0 f_H \mathrm{Sa}(2\pi f_H \tau)$.

$$g_{\tau'}(t) \longleftrightarrow \tau' \mathrm{Sa}(\frac{\omega\tau'}{2})$$

$$\tau' \mathrm{Sa}\frac{\tau' t}{2} \longleftrightarrow 2\pi G_{\tau'}(\omega), \quad \frac{\tau'}{2\pi}\mathrm{Sa}\frac{\tau' t}{2} \longleftrightarrow G_{\tau'}(\omega)$$

$$\tau' = 4\pi f_H,$$

$$\frac{n_0}{2} G_{\tau'}(\omega) \longleftrightarrow \frac{4\pi f_H}{2\pi}\frac{n_0}{2}\mathrm{Sa}(\frac{4\pi f_H \tau}{2}) = n_0 f_H \mathrm{Sa}\big(2\pi f_H \tau\big)$$

Narrowband Random Process

Definition: Assume that the frequency bandwidth of the random process is Δf, and the central frequency is f_c. If $\Delta f \ll f_c$, the random process is called narrow band random process.

Mathematical Description

- $\xi(t) = a_\xi(t)\cos[\omega_c t + \varphi_\xi(t)]$.
- $\xi(t) = \xi_c(t)\cos\omega_c t - \xi_s(t)\sin\omega_c t$
  - in-phase component $\xi_c(t) = a_\xi(t)\cos\varphi(t)$.
  - orthogonal component $\xi_s(t) = a_\xi(t)\sin\varphi(t)$.

Statistic Characteristics of $\xi_c(t)$ and $\xi_s(t)$

If $\xi(t)$ is a narrow band stationary Gaussian process with 0 mean, we have

- $\xi_c(t)$ and $\xi_s(t)$ are stationary Gaussian processes with 0 mean;
- $\sigma_\xi^2 = \sigma_{\xi_c}^2 = \sigma_{\xi_s}^2$;
- ξ_c and ξ_s are the same instant are uncorrelated.

Statistic Characteristics of $a_\xi(t)$ and $\varphi_\xi(t)$

If $\xi(t)$ is a narrow band stationary Gaussian process with 0 mean, we have

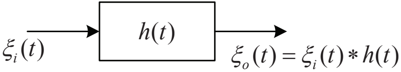
- $a_\xi(t)$: 1-dim distribution is Rayleigh distribution, i.e.,

$$f(a_\xi) = \frac{a_\xi}{\sigma_\xi^2} \exp\Big(-\frac{a_\xi^2}{2\sigma_\xi^2}\Big), \quad a_\xi \geq 0.$$

- $\varphi_\xi(t)$: 1-dim distribution is uniform distribution in $0 \sim 2\pi$.
- As for 1-dim distribution, $a_\xi(t)$ and $\varphi_\xi(t)$ are independent.

Random Process Transfer through Linear Systems

Assume that the system is physically realizable,
i.e.,$h(t) = 0, t < 0, \quad \int_{-\infty}^{\infty} |h(t)|dt < \infty$.



Assume $\xi_i(t)$ is stationary

- Mathematical expectation:
  $E(\xi_o(t)) = \int_0^{\infty} h(\tau)E[\xi_i(t-\tau)]d\tau = E(\xi_i(t))H(0)$.
- Autocorrelation:

$$\begin{aligned}
R_{\xi_o} &= E\left[\xi_o(t_1)\xi_o(t_1+\tau)\right] \\
&= E\int_0^{+\infty} h(u)\xi_i(t_1-u)du \int_0^{+\infty} h(v)\xi_i(t_1+\tau-v)dv \\
&= \int_0^{+\infty}\int_0^{+\infty} h(u)h(v)R_{\xi_i}(\tau+u-v)dudv = R_{\xi_o}(\tau).
\end{aligned}$$

- Power spectral density:

$$
\begin{aligned}
P_{\xi_o} &= \int_{-\infty}^{+\infty} R_{\xi_o}(\tau) e^{-j\omega\tau} d\tau \\
&= \int_{-\infty}^{+\infty} \int_{0}^{+\infty} \int_{0}^{+\infty} h(u)h(v)R_{\xi_i}(\tau + u - v) e^{-j\omega\tau} dudvd\tau \\
&= \int_{0}^{+\infty} h(u) e^{j\omega u} du \int_{0}^{+\infty} h(v) e^{-j\omega v} dv \int_{-\infty}^{+\infty} R_{\xi_i}(\tau) e^{-j\omega\tau} d\tau \\
&= |H(f)|^2 P_{\xi_i}.
\end{aligned}
$$

- If $\xi_i(t)$ is Gaussian, $\xi_o(t)$ is also Gaussian.

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