Ch5 Finite-Length Discrete Transforms

- (Optional) Let $\tilde{x}[n]$ be a periodic sequence with period *N*, i.e., $\tilde{x}[n] = \tilde{x}[n + lN]$, where *l* is any integer. The sequence $x[n]$ can be presented by Fourier series given by a weighted sum of periodic complex exponential sequences $\tilde{\psi}_k[n] = e^{j2\pi kn/N}$. Show that, unlike the Fourier series representation of a periodic continuous-time signal, the Fourier series representation of a periodic discrete-time sequence requires only *N* of the periodic complex exponential sequences $\tilde{\psi}_k[n]$, $k = 0, 1, \ldots, N-1$, and is of the form

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k] e^{j2\pi kn/N}$$

where the Fourier coefficients $\tilde{X}[k]$ are given by

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j2\pi kn/N}$$

Show that $\tilde{X}[k]$ is also a periodic sequence in *k* with a period *N*. The set of sequences represent the *discrete Fourier series* pair.

- (Optional) Let $x[n]$ be an aperiodic sequence with a DTFT $X(e^{j\omega})$. Define

$$\tilde{X}[k] = X(e^{j\omega})\Big|_{\omega=2\pi k/N} = X(e^{j2\pi k/N}), \quad -\infty < k < \infty$$

Show that $\tilde{X}[k]$ is a periodic sequence in *k* with a period *N*. Let $\tilde{X}[k]$ be the discrete Fourier series coefficients, defined in optional question before, of the periodic sequence $\tilde{x}[n]$. Shown with the equations in optional question before, that

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n + rN]$$

5.1 Determine the *N*-point DFTs of the following length-*N* sequences defined for $0 \le n \le N-1$:

a) $x_1[n] = \cos(2\pi n / N)$

b) $x_2[n] = \sin^2(2\pi n / N)$

c) $x_3[n] = \alpha^n$

d) $x_4[n] = \begin{cases} 4, & \text{for } n \text{ even} \\ -2, & \text{for } n \text{ odd} \end{cases}$

5.2 Determine the N-point DFT $X[k]$ of N-point sequence $x[n] = \cos(\omega_0 n)$, $0 \le n \le N-1$, for $\omega_0 \ne 2\pi r / N$, where r is an integer in the range $0 < r < N-1$.

5.3 Consider a length-N sequence $x[n], 0 \le n \le N-1$, with N even. Define two sequences of length-N/2 given by:

$$g[n] = (x[n] + x[\frac{N}{2} + n]), \quad h[n] = (x[n] - x[\frac{N}{2} + n])W_N^n, \qquad 0 \le n \le \frac{N}{2} - 1.$$

If $G[k]$ and $H[k]$, $0 \le k \le \frac{N}{2} - 1$, denote the N/2-point DFT of $g[n]$ and $h[n]$, respectively. Determine the N-point DFT $X[k], 0 \le k \le N-1$, of $x[n]$, from these two N/2-point DFTs.

5.4 Let $X[k]$ denote the N-point DFT of a length-N sequence $x[n]$, with N even. Define two length-N/2 sequences given by:

$$g[n] = \frac{1}{2}(x[2n] + x[2n+1]), \qquad h[n] = \frac{1}{2}(x[2n] - x[2n+1]), \quad 0 \le n \le \frac{N}{2} - 1.$$

If $G[k]$ and $H[k]$, $0 \le k \le \frac{N}{2} - 1$, denote the N/2-point DFT of $g[n]$ and $h[n]$, respectively. Determine the N-point DFT $X[k]$ from these two N/2-point DFTs.

- (Optional) Let $X[k], 0 \le k \le N-1$, denote the N-point DFT of a length-N sequence $x[n]$, with N even. Define two sequences of length-N/2 given by:

$$g[n] = a_1 x[2n] + a_2 x[2n+1], \quad h[n] = (a_3 x[2n] + a_4 x[2n+1]), \quad 0 \le n \le \frac{N}{2} - 1.$$

where $a_1 a_4 \ne a_2 a_3$. If $G[k]$ and $H[k], 0 \le k \le \frac{N}{2} - 1$, denote N/2-point DFTs of $g[n]$ and $h[n]$, respectively. Determine the N-point DFT $X[k]$ from these two N/2-point DFTs.

5.5

a) Let $x[n], 0 \le n \le N-1$, be a length-N sequence with an N-point DFT given by $X[k]$, $0 \le k \le N-1$. Determine the $2N$-point DFTs of the following length-$2N$ sequences in terms of $X[k]$.

i. $$g[n] = \begin{cases} x[n], & 0 \le n \le N-1 \\ 0, & N \le n \le 2N-1 \end{cases}$$

ii. $$h[n] = \begin{cases} 0, & 0 \le n \le N-1 \\ x[n-N], & N \le n \le 2N-1 \end{cases}$$

b) Let $G[k]$ and $H[k]$, $0 \le k \le 2N-1$, denote, respectively the $2N$-point DFTs of the length-$2N$ sequences $g[n]$ and $h[n]$ in Question a). Define a new length-$2N$ sequence by $y[n] = g[n] + h[n]$, with a $2N$-point DFT $Y[k]$ $0 \le k \le 2N-1$. Develop the relation between $Y[k]$, $H[k], G[k]$ and $X[k]$.

- Let $x[n]$ be a length-N sequence with $X[k]$ denoting its N-point DFT. We present the DFT operation as $X[k] = \mathscr{F}\{x[n]\}$. Determine the sequence $y[n]$ obtained by applying the DFT operation 4 times to $x[n]$, i.e.,

$$y[n] = \mathscr{F}\{\mathscr{F}\{\mathscr{F}\{\mathscr{F}\{x[n]\}\}\}\}$$

5.6 Consider a length-N sequence $x[n]$, $0 \le n \le N-1$, with an N-point DFT $X[k]$, $0 \le k \le N-1$.

a) Let $Y[k]$ denote the MN-point DFT of the sequence $x[n]$ appended with $(M-1)N$ zeros. Show that the N-point DFT $X[k]$ can be simply obtained from $Y[k]$ as follows:

$$X[k] = Y[kM], \quad 0 \le k \le N-1$$

b) Define a sequence $y[n]$ of length $N/3$, , given by:

$$y[n] = x[3n], \quad 0 \le n \le N/3 - 1$$

Express the $N/3$-point DFT $Y[k]$ in terms of $X[k]$.

c) Define a sequence $y[n]$ of length LN, $0 \le n \le LN-1$, given by:

$$y[n] = \begin{cases} x[n/L], & n = 0, L, 2L, \ldots, (N-1)L \\ 0, & \text{otherwise} \end{cases}$$

where L is a positive integer. Express the NL-point DFT $Y[k]$ in terms of $X[k]$.

5.7 Let $x[n]$, $0 \le n \le N-1$ be a length-N sequence with an N-point DFT $X[k]$, denoting its N-point DFT. Define a length-$3N$ sequence by

$$y[n] = \begin{cases} x[n], & 0 \le n \le N-1 \\ 0, & N \le n \le 3N-1 \end{cases}$$

with $Y[k]$, $0 \le k \le 3N-1$, denoting its $3N$-point DFT. Let $W[\ell] = Y[3\ell+2]$, $0 \le \ell \le N-1$, with $w[n]$ denoting its N-point DFT. Express $w[n]$ in terms of $x[n]$.

- Let $x[n]$, $0 \le n \le N-1$ be a even length sequence with an N-point DFT $X[k]$, $0 \le k \le N-1$. If $x[2m] = 0$ for $0 \le m \le \frac{N}{2} - 1$, show that $x[n] = -x[<n + \frac{N}{2}>_N]$.

5.8 Let $x[n]$, $0 \le n \le N-1$ be a even length sequence with an N-point DFT $X[k]$, $0 \le k \le N-1$. Determine the N-point DFTs of the following N-point sequences in terms of $X[k]$.

a) $u[n] = x[n] - x[\left\langle n - \frac{N}{2} \right\rangle_N]$

b) $v[n] = x[n] + x[\left\langle n - \frac{N}{2} \right\rangle_N]$

c) $w[n] = (-1)^n x[n]$

- (Optional) Consider a rational discrete-time Fourier transform $X(e^{j\omega})$ with real

coefficients of the form of

$$X(e^{j\omega}) = \frac{P(e^{j\omega})}{D(e^{j\omega})} = \frac{p_0 + p_1 e^{-j\omega} + \cdots + p_{M-1} e^{-j\omega(M-1)}}{d_0 + d_1 e^{-j\omega} + \cdots + d_{N-1} e^{-j\omega(N-1)}}$$

Let $P[k]$ denote the *M*-point DFT of the numerator coefficients $\{p_i\}$ and $D[k]$ denote the *N*-point DFT of the denominate coefficients $\{d_i\}$. Determine the exact expressions of the DTFT $X(e^{j\omega})$ for *M*=*N*=4 if the 4-point DFTs of its numerator and denominator coefficients are given by

$$P[k] = \{3.5,\ -0.5 - j9.5,\ 2.5,\ -0.5 + j9.5\}$$

$$D[k] = \{17,\ 7.4 + j12,\ 17.8,\ 7.4 - j12\}$$

Verify your result using MATLAB.

5.9 Let $x[n]$, $0 \le n \le N-1$ be a length-*N* real sequence with an *N*-point DFT $X[k]$, $0 \le k \le N-1$.

a) Show that $X[\langle N-k \rangle_N] = X^*[k]$;

b) Show that $X[0]$ is real;

c) If *N* is even, show that $X[N/2]$ is real.

5.10 Without computing the DFT, determine which one of the following length-9 sequences defined for $0 \le n \le 8$ has a real-valued 9-point DFT and which one has an imaginary-valued 9-point DFT. Justify your answer.

a) $x_1[n] = \{5,\ -9,\ 4,\ 7,\ -8,\ -8,\ 7,\ 4,\ -9\}$,

b) $x_2[n] = \{0,\ -4,\ 3,\ 7,\ -5,\ 5,\ -7,\ -3,\ 4\}$

5.11 Let $G[k]$ and $H[k]$, $0 \le k \le 7$ denote the 8-point DFTs of two length-8 sequences $g[n]$ and $h[n]$, $0 \le n \le 7$, respectively.

a) If $G[k] = \{3 + j4,\ 2 - j7,\ -4 + j,\ 5 - j2,\ 5,\ 4 + j3,\ 4 - j6,\ -3 + j2\}$ and $h[n] = g[\langle n-3 \rangle_8]$, determine $H[k]$ without forming $h[n]$ and computing its DFT.

b) If $g[n] = \{-1 - j7,\ 3 + j,\ 2 + j7,\ 2 + j2,\ -8 + j2,\ 4 - j,\ -1 + j3,\ j1.5\}$

and $H[k] = G[\langle k+5 \rangle_8]$, determine $h[n]$ without computing the DFT $G[k]$, forming $H[k]$ and then finding its inverse DFT.

- (Optional) Consider two length-N real-valued sequences $x[n]$ and $y[n]$ defined for $0 \le n \le N-1$, with N-point DFTs $X[k]$ and $Y[k]$, $0 \le k \le N-1$, respectively. The *circular correlation* of $x[n]$ and $y[n]$ is given by

$$r_{xy}[\ell] = \sum_{n=0}^{N-1} x[n]x[\langle \ell + n \rangle_N], \quad 0 \le \ell \le N-1$$

Express the DFT of $r_{xy}[\ell]$ in terms of $X[k]$ and $Y[k]$.

5.12 Let $x[n]$, $0 \le n \le N-1$ be a length-N sequence with an MN-point DFT $X[k]$, $0 \le k \le MN-1$. Define

$$y[n] = x[\langle n \rangle_N], \quad 0 \le n \le MN-1$$

How would you compute the MN-point DFT $Y[k]$ of $y[n]$ knowing only $X[k]$?

5.13 Consider the length-10 sequence, defined for $0 \le n \le 9$,

$$x[n] = \{-3, \quad 5, \quad -7, \quad 8 \quad 2, \quad -8, \quad -4, \quad 1, \quad -9, \quad 9\}$$

with a 10-point DFT given by $X[k], 0 \le k \le 9$. Evaluate the following functions of $X[k]$ without computing the DFT:

a) $X[0]$,

b) $X[5]$,

c) $\sum_{k=0}^{9} X[k]$,

d) $\sum_{k=0}^{9} e^{-j3\pi k/5} X[k]$,

e) $\sum_{k=0}^{9} |X[k]|^2$.

5.14 Let $X[k]$, $0 \le k \le 11$, be a 12-point DFT of a length-12 real sequence $x[n]$ with the first seven samples of $X[k]$ given by:

$$X[k] = \{11, \quad 8 - j2, \quad -1 + j4, \quad -6 + j3, \quad 3 + j2, \quad 2 - j4, \quad 4\}, \quad 0 \le k \le 6$$

Determine remaining samples of $X[k]$. Evaluate the following function of $x[n]$ without computing the IDFT of $X[k]$:

a) $x[0]$,

b) $x[6]$,

c) $\sum_{n=0}^{11} x[n]$,

d) $\sum_{n=0}^{11} e^{j2\pi n/3} x[n]$,

e) $\sum_{n=0}^{11} |x[n]|^2$

5.15 The following 5 samples of 9-point DFT $X[k]$ of a real length-9 sequence $x[n]$ are given by: $X[0] = 11$, $X[2] = 1.2 - j2.3$, $X[3] = -7.2 - j4.1$, $X[5] = -3.1 + j8.2$, $X[8] = 4.5 + j1.6$. Determine the remaining 4 samples of the DFT.

5.16 The first 7 samples of a length-12 real sequence $x[n]$ with an imaginary-valued 12-point DFT $X[k]$ are given by: $x[0] = 0$, $x[1] = 0.7$, $x[2] = -3.25$, $x[3] = 4.1$, $x[4] = 2.87$, $x[5] = -9.3$ and $x[6] = 0$. Determine the remaining 5 samples of $x[n]$.

- (Optional) 158-point DFT $X[k]$ of a real-valued sequence $x[n]$ has the following DFT samples: $X[0] = 31$, $X[15] = 4.13 - j8.27$, $X[k_1] = 6.1 + j2.8$, $X[41] = -3.15 - j2.04$, $X[k_2] = -7.3 - j9.5$, $X[80] = 9.08$, $X[119] = 6.1 - j2.8$, $X[k_3] = 4.13 + j8.27$, $X[151] = -7.3 + j9.5$, and $X[k_4] = -3.15 + j2.04$. Remaining DFT samples are assumed to

be of zero value.

a) Determine the values of the indices k_1, k_2, k_3 and k_4.

b) What is the dc value of $\{x[n]\}$?

c) What is the energy of $\{x[n]\}$?

5.17 Let $X(e^{j\omega})$ denote the DTFT of the length-9 sequence

$$x[n] = \{1, \quad -3, \quad 4, \quad -5, \quad 7, \quad -5, \quad 4, \quad -3, \quad 1\}$$

a) For the DFT sequence $X_1[k]$, obtained by sampling $X(e^{j\omega})$ at uniformly intervals of $\pi/6$ starting from $\omega = 0$, determine the IDFT $x_1[n]$ of $X_1[k]$ without computing $X(e^{j\omega})$ and $X_1[k]$. Can you recover $x[n]$ from $x_1[n]$?

b) For the DFT sequence $X_2[k]$, obtained by sampling $X(e^{j\omega})$ at uniformly intervals of $\pi/4$ starting from $\omega = 0$, determine the IDFT $x_2[n]$ of $X_2[k]$ without computing $X(e^{j\omega})$ and $X_2[k]$. Can you recover $x[n]$ from $x_2[n]$?

- (Optional) Let $x[n]$ and $h[n]$ be two length-51 sequences defined for $0 \le n \le 50$. It is known that $h[n] = 0$ for $0 \le n \le 16$ and $37 \le n \le 50$. Denote the 51-point circular convolution of these sequences as $y_C[n]$ and the linear convolution as $y_L[n]$. Determine the range of n for which $y_L[n] = y_C[n]$.

5.18

a) Let $g[n]$ and $h[n]$ be two finite-length sequences of length 6 each. If $y_L[n]$ and $y_C[n]$ denote the linear and 6-point circular convolutions of $g[n]$ and $h[n]$, respectively, develop a method to determine $y_C[n]$ in terms of $y_L[n]$.

b) Let $y_C[n]$ denote the 6-point circular convolutions of two length-6 sequences

$$x[n] = \{-3, \quad 0, \quad 7, \quad 4, \quad -5, \quad 8\}$$

$$h[n] = \{7, \quad -2, \quad 4, \quad -5, \quad 0, \quad 6\}$$

Determine the $y_L[n]$ obtained by a linear convolution of $x[n]$ and $h[n]$. Determine the sample value $y_C[3]$ using the method developed in Part a) without carrying out the circular convolution.

- (Optional) Show that the circular convolution is
  c) Commutative;
  d) Associative.

5.19 A length-9 sequence is given by $x[n] = \{3, \quad 5, \quad 1, \quad 4, \quad -3, \quad 5, \quad -2, \quad -2, \quad 4\}$, $0 \le n \le 8$, with an 9-point DFT given by $X[k], 0 \le k \le 8$. Without computing the IDFT, determine the sequence $y[n]$ whose 9-point DFT is given by $Y[k] = W_3^{-2k} X[k]$.

5.20 The first 5 samples of 9-point DFT $X[k]$ $0 \le k \le 8$ is given by

$$X[k] = \{15, \quad 7 - j6, \quad 6 - j2, \quad j8, \quad -6 - j11\},$$

Without computing the IDFT, determine the 9-point DFT $Y[k]$ of the length-9 sequence $y[n] = e^{j2\pi n/3} x[n]$

5.21 Consider the two finite-length sequences $h[n] = \{4, \quad -3, \quad 1, \quad -4\}$, $0 \le n \le 3$, and $g[n] = \{-3, \quad 2, \quad 5\}, 0 \le n \le 2$.

a) Determine $y_L[n] = g[n] * h[n]$;

b) Extend $g[n]$ to a length-4 sequence $g_e[n]$ by zero-padding and compute $y_C[n] = g[n] \circledast_4 h[n]$;

c) Determine $y_C[n]$ using the DFT-based approach;

d) Extend $g[n]$ and $h[n]$ to length-6 sequences by zero-padding and compute the 6-point circular convolution $y[n]$ of the extended sequences. Is $y[n]$ the same as

$y_L[n]$ determined in Part a).

- (Optional) Let $x[n]$, $0 \le n \le N-1$ be a length-N sequence with an N-point DFT $X[k]$, $0 \le k \le N-1$.

  a) If $x[n]$ is a symmetric sequence satisfying the condition $x[n] = x[\langle N-1-n \rangle_N]$, show that $X[N/2] = 0$ for N even.

  b) If $x[n]$ is an antisymmetric sequence satisfying the condition $x[n] = -x[\langle N-1-n \rangle_N]$, show that $X[0] = 0$.

  c) If $x[n]$ is a sequence satisfying the condition $x[n] = -x[\langle n+M \rangle_N]$ with N=2M, show that $X[2\ell] = 0$ for $\ell = 0,\ 1,\ \ldots,\ M-1$.

- (Optional) Consider two real, symmetric length-N sequences, $x[n]$ and $y[n]$, $0 \le n \le N-1$ with N even. Define the length-N/2 sequences:

$$x_0[n] = x[2n+1] + x[2n] \qquad x_1[n] = x[2n+1] - x[2n]$$

$$y_0[n] = y[2n+1] + y[2n] \qquad y_1[n] = y[2n+1] - y[2n]$$

  where $0 \le n \le \frac{N}{2} - 1$. It can be easily shown that $x_0[n]$ and $y_0[n]$ are real, symmetric sequences of length-N/2 each. Likewise, $x_1[n]$ and $y_1[n]$ are real, antisymmetric sequences of length-N/2 each. Denote the $\frac{N}{2}$-point DFTs of $x_0[n], y_0[n], x_1[n]$ and $y_1[n]$ by $X_0[k], Y_0[k], X_1[k]$ and $Y_1[k]$, respectively. Define a length-$\frac{N}{2}$ sequence:

$$u[n] = x_0[n] + y_1[n] + j(x_1[n] + y_0[n])$$

  Determine $X_0[k], Y_0[k], X_1[k]$ and $Y_1[k]$ in terms of the $\frac{N}{2}$-point DFT $U[k]$.

- (Optional) Let $x[n]$, $0 \le n \le N-1$ be a length-N sequence with an N-point DFT $X[k]$, $0 \le k \le N-1$.

  e) Show that if N is even and $x[n] = -x[\langle n + \frac{N}{2} \rangle_N]$ for all n, then $X[k] = 0$ for k even;

f) Show that if N is an integer of 4 and $x[n] = -x[\langle n + \frac{N}{2} \rangle_N]$ for all n, then $X[k] = 0$ for k=4ℓ, $0 \le \ell \le \frac{N}{4} - 1$

5.22 Let $x[n]$, $0 \le n \le N-1$ be a length-N sequence with an N-point DFT $X[k], 0 \le k \le N-1$. Determine the N-point DFTs of the following length-N sequences in terms of $X[k]$.

a) $w[n] = \alpha x[\langle n - m_1 \rangle_N] + \beta x[\langle n - m_2 \rangle_N]$, where m_1 and m_2 are positive integers less than N,

b) $g[n] = \begin{cases} x[n], & \text{for } k \text{ even} \\ 0, & \text{for } k \text{ odd} \end{cases}$,

c) $y[n] = x[n] \circledN x[n]$

5.23 Let $x[n]$, $0 \le n \le N-1$ be a length-N sequence with an N-point DFT $X[k], 0 \le k \le N-1$. Determine the N-point inverse DFTs of the following length-N DFTs in terms of $x[n]$.

a) $W[k] = \alpha X[\langle k - m_1 \rangle_N] + \beta X[\langle k - m_2 \rangle_N]$, where m_1 and m_2 are positive integers less than N,

b) $G[k] = \begin{cases} X[k], & \text{for } k \text{ even} \\ 0, & \text{for } k \text{ odd} \end{cases}$,

c) $Y[k] = X[k] \circledN X[k]$

5.24 The first seven samples of the 12-point DFT $H[k], 0 \le k \le 11$, of a length12 real sequence $h[n]$, $0 \le n \le 11$, are given by

$H[k] = \{4, \quad 17.19 + j1.46, \quad -9 + j3.46, \quad -9 + j5, \quad 1 + j24.25, \quad 6.8 - j5.46, \quad 6\}$,

$0 \le k \le 6$. Determine the 12-point DFT $G[k]$ of the length-12 sequence $g[n] = h[< n - 17 >_N]$ without computing $h[n]$, forming the sequence $g[n]$, and then taking its DFT.

5.25 Let $x[n]$, $0 \le n \le N-1$ be a length-8 sequence given by

$$\{x[n]\} = \{2, \quad 4, \quad 6, \quad 8, \quad 1, \quad 3, \quad 5, \quad 7\}, \ 0 \le n \le 7$$

with $X(e^{j\omega})$ denoting its DTFT. Define $Y[k] = X(e^{j2k\pi/5})$, $0 \le k \le 4$, with $y[n]$ denoting its 5-point IDFT. Determine $y[n]$ without computing $Y[k]$ of $y[n]$ in terms of $x[n]$.