Nonlinear Analysis: Real World Applications 14 (2013) 1511–1526

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis: Real World Applications

journal homepage: www.elsevier.com/locate/nonrwa

Stability of traveling waves in a monostable delayed system without quasi-monotonicity

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ARTICLE INFO

Article history:
Received 8 November 2011
Accepted 16 October 2012

Keywords:
Traveling waves
Weighted energy method
Stability

ABSTRACT

This paper is concerned with traveling waves of a monostable reaction–diffusion system with delay and without quasi-monotonicity. When the initial perturbation around the traveling wave is suitably small in a weighted norm, the exponential stability of all traveling wave solutions for the system with delay is proved by the weighted energy method.

1. Introduction

Traveling wave solutions of nonlinear reaction–diffusion equations and systems have been extensively and intensively investigated due to their important role in a variety of physical, biological, and epidemic problems; see [1–22]. The investigation of the stability of traveling wave solutions is one of the important and difficult aspects in the theory of traveling waves. In particular, the loss of (quasi-)monotonicity and incorporation of (nonlocal) delay into many realistic models in applications may result in much difficulty, because the time delay and spatial non-locality lead to some changes of dynamics for equations and systems, it is no longer suitable to use the frequent methods and theory for solving the problems of traveling wave solutions in (quasi-)monotone reaction–diffusion equations and systems or without delay.

Regarding the monotone traveling wave problems for some scalar reaction–diffusion equations and systems with delay, much has been done concerning the stability of wavefronts by using the spectral analysis method, the squeezing technique, the weighted energy method, the combination of the comparison principle and the weighted energy method, and so on. For example, one pioneering work from Schaaf [23] established the local stability results of traveling wave solutions for a scalar equation with delay, Fisher–KPP nonlinearity and one equilibrium $u_- = 0$ is a stable node, by the spectral analysis method. In [24], by using the squeezing technique (please refer to [25–29]), we studied the globally asymptotic exponential stability of traveling fronts for a bistable quasi-monotone system with delay. Recently, with the help of the method of combination of the comparison principle and the weighted energy method (please refer to [30–36]), we [37] established the exponential stability of monotone traveling wave solutions for the large initial perturbation and large wave speed of the following monostable reaction–diffusion system with delay

$$\begin{cases} \dfrac{\partial}{\partial t}u_1(t,x) = \dfrac{\partial^2}{\partial x^2}u_1(t,x) - u_1(t,x) + u_2(t,x), \\ \dfrac{\partial}{\partial t}u_2(t,x) = -\beta u_2(t,x) + g(u_1(t-\tau,x)) \end{cases} \tag{1.1}$$

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doi:10.1016/j.nonrwa.2012.10.015

when g is the monotone Nicholson's birth function, i.e. $g(u) = pue^{-au}(1 < \frac{p}{\beta} \le e)$ is monotone, and the development of the epidemic model can be referred to [5,21,22,38–40].

However, on the traveling wave solutions of reaction–diffusion equations and systems without quasi-monotonicity, it seems that little has been done regarding the existence of traveling wave solutions, not to mention the study of the stability of traveling wave solutions. For delayed scalar reaction–diffusion equations without quasi-monotonicity, some existence results of traveling wave solutions have been obtained in Huang and Zou [41] and Wu and Zou [42] by using the idea of the so-called exponential ordering for delayed differential equations. But these results are valid only for small values of the delay. Fortunately, Ma [43] complement the defect by the idea of auxiliary equations and Schauder's fixed point theorem applied in Ma [44]. His existence result is valid for all values of the delay, in contrast to the results in [41,42]. After that, using the same method, Wu and Li et al. [17,45] established the existence of traveling wave solutions for some complicated non-local reaction–diffusion equations with delay and non-local diffusion term equations with delay, respectively. In particular, Wu [46] applied the method to solve the existence of traveling wave solutions of a class of non-monotone integral equations. As an application of this result, the existence of traveling wave solutions of the following epidemic system with distributed delay

$$\begin{cases} \dfrac{\partial}{\partial t}u_1(t,x) = \dfrac{\partial^2}{\partial x^2}u_1(t,x) - u_1(t,x) + u_2(t,y), \\ \dfrac{\partial}{\partial t}u_2(t,x) = -\beta u_2(t,x) + \displaystyle\int_0^{\infty} g(u_1(t-s,x))P(ds), \end{cases} \tag{1.2}$$

has been obtained. It is easy to see that if $P(\cdot)$ is Dirac function $\delta(\cdot)$, (1.2) is the system (1.1). On the other hand, for scalar equations with local delay but without quasi-monotonicity, Wu [47] established the stability results of traveling wave solutions by using the weighted energy method derived from Mei's [48,49] idea. However, to the best of our knowledge, there are no stability results of traveling wave solutions for delayed reaction–diffusion systems without quasi-monotonicity up until now. Based on this fact, in this paper, we want to continue to study the stability of traveling wave solutions of the delayed system (1.1) when $g(u_1)$ is not satisfied with monotonicity. Therefore the investigated system (1.1) does not satisfy the so-called quasi-monotone condition and the comparison principle is not applicable for the system. Naturally, the frequently used methods for stability of traveling wave solutions, such as the squeezing technique, the method of combination of the comparison principle and the weighted energy method are not valid. In addition, the spectrum analysis is very complicated for delayed systems. Fortunately the weighted energy method does not need the comparison principle to hold, but it is not clear whether this method can also be used to solve the stability of traveling wave solutions of these non-monotone systems. As a result, in this paper we applied the weighted energy method to solve the stability of traveling wave solutions for a monostable system with delay and without quasi-monotonicity.

The rest of this paper is organized as follows. In Section 2, we introduce some preliminaries and state our stability result. In Section 3, we prove our main result on the exponential stability of traveling waves.

2. Preliminaries and main result

Notations. Throughout this paper, $C > 0$ denotes a generic constant, $C_i > 0$ $(i = 1, 2, \ldots)$ represents a specific constant. Let I be an interval. $L^2(I)$ is the space of the square integrable functions defined on I, and $H^k(I)(k \ge 0)$ is the Sobolev space of the L^2-functions $f(x)$ defined on the interval I whose derivatives $\frac{d^i}{dx^i}f$ $(i = 1, 2, \ldots, k)$ also belong to $L^2(I)$. $L^2_w(I)$ denotes the weighted L^2-space with a weight function $w(x) > 0$ and its norm is defined by $\|f\|_{L^2_w} = \left(\int_I w(x)\,|f(x)|^2\,dx\right)^{\frac{1}{2}}$, $H^k_w(I)$ is the weighted Sobolev space with the norm given by

$$\|f\|_{H^k_w} = \left(\sum_{i=0}^{k}\int_I w(x)\left|\frac{d^i}{dx^i}f(x)\right|^2 dx\right)^{\frac{1}{2}}.$$

Let $T > 0$ be a number and $\mathcal{B}$ be a Banach space. We denote by $C([0,T];\mathcal{B})$ the space of the $\mathcal{B}$-valued continuous functions on $[0,T]$. $L^2([0,T];\mathcal{B})$ is the space of the $\mathcal{B}$-valued L^2-functions on $[0,T]$. The corresponding spaces of $\mathcal{B}$-valued functions on $[0,\infty)$ are defined similarly.

In addition, throughout this paper, we assume that $\beta > \frac{1}{2}$ and (1.1) satisfies the initial conditions

$$\begin{cases} u_1(s,x) = u_{10}(s,x), \quad s \in [-\tau, 0],\ x \in \mathbb{R}, \\ u_2(0,x) = u_{20}(x), \quad x \in \mathbb{R}. \end{cases} \tag{2.1}$$

We also need the following assumptions for the sake of proving the existence of traveling wave solutions (see [46]):

(C_1) $g(0) = \beta K - g(K) = 0$ for some $K > 0$, $g'(0) > \beta$, and there exists a $\nu \in (0,1]$ such that

$$\limsup_{u\to 0+}\left[g'(0) - \frac{g(u)}{u}\right]u^{-\nu} < +\infty;$$

(C_2) $\min\{k^* u, K^+\} \geq \frac{1}{\beta} g(u) > 0$ for $\forall u \in (0, K^+]$, and g is Lipschitz continuous on $[0, K^+]$ for some $K^+ \geq K$, where $k^* = \frac{g'(0)}{\beta}$;

(C_3) One of the following assumptions holds:

(i) $K \geq \frac{1}{\beta} g(u) > u$, for $\forall u \in (0, K)$;

(ii) $u < \frac{1}{\beta} g(u) < 2K - u$, for $u \in (K^-, K)$ and $u > \frac{1}{\beta} g(u) > 2K - u$, for $u \in (K, K^+)$, where $K^- = k^* \inf_{\eta \in [0, K^+]} \{g(\eta) : \frac{1}{\beta} g(\eta) \leq \eta\}$;

(iii) $u < \frac{1}{\beta} g(u)$, for $u \in [K^-, K)$; $u > \frac{1}{\beta} g(u)$, for $u \in (K, K^+]$ and there is no pair $0 < \gamma_1 < K < \gamma_2 \leq K^+$ such that $\gamma_1 = \frac{1}{\beta} g(\gamma_2)$, $\gamma_2 = \frac{1}{\beta} g(\gamma_1)$.

Notice that system (1.1) has two constant equilibria $\mathbf{u}_- = (u_{1-}, u_{2-}) = (0, 0)$ and $\mathbf{u}_+ = (u_{1+}, u_{2+}) = (K, K_0)$, where $K_0 = \frac{g(K)}{\beta}$ and $K, K_0 > 0$. We are interested in traveling wave solutions $\mathbf{u}$ of (1.2) that connect $\mathbf{u}_-$ with $\mathbf{u}_+$. A traveling wave solution of system (1.1) connecting with $\mathbf{u}_-$ and $\mathbf{u}_+$ is a solution $\mathbf{u} = (u_1(t, x), u_2(t, x)) = (\phi_1(\xi), \phi_2(\xi))$, $\xi = x + ct$, satisfying the following ordinary differential system

$$\begin{cases} c\phi_1'(\xi) - \phi_1''(\xi) + \phi_1(\xi) = \phi_2(\xi), \\ c\phi_2'(\xi) + \beta\phi_2(\xi) = g(\phi_1(\xi - c\tau)), \\ \phi_1(\pm\infty) = u_{1\pm}, \qquad \phi_2(\pm\infty) = u_{2\pm}. \end{cases} \tag{2.2}$$

Obviously, when $P(\cdot)$ is Dirac function $\delta(\cdot)$, (1.2) is the system (1.1), therefore, the existence of traveling wave solutions of (1.1) is guaranteed by the following Proposition 2.1. Wu [46] proved the existence of traveling wave solutions of (1.1) with profile $(\phi_1(\xi), \phi_2(\xi))$ by the idea of auxiliary equations and Schauder's fixed point theorem.

Proposition 2.1 (*Existence of Traveling Waves*)**.** *Assume that* (C_1)–(C_2) *hold. Then there exists* $c_* > 0$ *such that*

(i) *for every* $c > c_*$, (1.2) *admits a traveling wave solution* $\mathbf{\Phi}(\xi) =: (\phi_1(\xi), \phi_2(\xi))$ *satisfying* $\phi_i(\xi) = O\left(e^{\Lambda_1(c)\xi}\right)$ *when* $\xi \to -\infty$, $i = 1, 2$, $\phi_1 \in C(\mathbb{R}, [0, K^+])$ *and*

$$0 < K^- \leq \liminf_{\xi \to +\infty} \phi_1(\xi) \leq \limsup_{\xi \to +\infty} \phi_1(\xi) \leq K^+,$$

$$0 < \liminf_{\xi \to +\infty} \phi_2(\xi) \leq \limsup_{\xi \to +\infty} \phi_2(\xi) \leq K^+.$$

If, in addition, (C_3) *also holds, then* $\mathbf{\Phi}(+\infty) = (K, K_0)$, *where* $K_0 = \frac{g(K)}{\beta}$ *and* $\Lambda_1(c)$ *is the smallest solution such that the linearized characteristic equation at* $(0, 0)$ *of* (1.2) *has solutions;*

(ii) *for* $c = c_*$, (1.2) *admits a traveling wave solution with the wave speed* c_**;*

(iii) *for* $\forall c \in (0, c_*)$, (1.2) *admits no such wave solution with the wave speed* c.

For some kind of need for proof, we assume g' is bounded on $[0, +\infty)$, and let

$$L = \max_{u \in [0, K^+]} |g'(u)|, \qquad \tilde{L} = \max_{u \in [0, +\infty)} |g'(u)|,$$

$$L_1 = \frac{1}{\beta} \max_{u \in [0, K^+]} |g''(u)| \max_{u \in [0, K^+]} g(u), \qquad L_2 = \frac{1}{2} \max_{u \in [-\sigma K^+, (\sigma+1)K^+]} |g''(u)|,$$

where $\sigma \geq 1$, and the definition of g can be referred to in the extension $\bar{g}$ for g in the following.

Lemma 2.2. *For a given traveling wave solution* $(\phi_1(\xi), \phi_2(\xi))$ *of* (1.1) *with speed* $c > c_*$, *there holds* $\lim_{\xi \to +\infty} \phi_1'(\xi) = 0$ *and* $|\phi_1'(\xi)| \leq \frac{1}{\beta} \max_{u \in [0, K^+]} g(u)$.

Proof. From the first equation of (2.2), we have

$$\phi_1(\xi) = \frac{1}{\lambda_2 - \lambda_1} \left[\int_{-\infty}^{\xi} e^{\lambda_1(\xi - s)} \phi_2(s) ds + \int_{\xi}^{+\infty} e^{\lambda_2(\xi - s)} \phi_2(s) ds \right].$$

Then

$$\phi_1'(\xi) = \frac{1}{\lambda_2 - \lambda_1} \left[\lambda_1 \int_{-\infty}^{\xi} e^{\lambda_1(\xi - s)} \phi_2(s) ds + \lambda_2 \int_{\xi}^{+\infty} e^{\lambda_2(\xi - s)} \phi_2(s) ds \right],$$

where

$$\lambda_1 = \frac{c - \sqrt{c^2 + 4}}{2} < 0, \qquad \lambda_2 = \frac{c + \sqrt{c^2 + 4}}{2} > 0$$

are two solutions of $c\lambda - \lambda^2 + 1 = 0$.

Applying the (General) L. Hospital's rule to the expression of $\phi_1'(\xi)$, we have $\lim_{\xi\to+\infty}\phi_1'(\xi)=0$. On the other hand,

$$\phi_2(\xi)=\frac{1}{c}\int_{-\infty}^{\xi} e^{-\frac{\beta}{c}(\xi-s)} g(\phi_1(s-c\tau))ds,$$

therefore, $\max_{\xi\in\mathbb{R}}\phi_2(\xi)\le\frac{1}{\beta}\max_{u\in[0,K^+]}g(u)$. Notice that $\lambda_2-\lambda_1\ge 2$, thus we get $|\phi_1'(\xi)|\le\max_{\xi\in\mathbb{R}}\phi_2(\xi)\le\frac{1}{\beta}\max_{u\in[0,K^+]}g(u)$. This completes the proof. □

Lemma 2.3. *If $g\in C^2([0,K^+],\mathbb{R})$ and $|g'(K)|<1$, then there exists $\xi^*\in\mathbb{R}$ such that, for any $\xi\ge\xi^*$,*

$$|g'(\phi_1(\xi))|<|g'(K)|+\epsilon,\qquad |g'(\phi_1(\xi-c\tau))|<|g'(K)|+\frac{\epsilon}{4},$$

and

$$|g'(\phi_1(\xi-c\tau))|+|g''(\phi_1(\xi-c\tau))\phi_1'(\xi-c\tau)|<|g'(K)|+\frac{\epsilon}{2},$$

where $\epsilon=\min\{\frac{1-|g'(K)|}{2},2\beta-1\}$.

Proof. By Lemma 2.2 and the fact that $g\in C^2([0,K^+],\mathbb{R})$, the results are obvious. □

In order to investigate the stability of traveling wave solutions of (1.1), we need to establish the global existence and uniqueness result of solution (u_1,u_2) to the Cauchy problem (1.1) and (2.1).

We first establish the non-negativity of all global solutions.

Proposition 2.4 (*Non-Negativity*)**.** *Let $(u_1(t,x),u_2(t,x))$ be the solution of (1.1) and (2.1) in $(0,+\infty)\times\mathbb{R}$. If $u_{10}(s,x)\ge 0$ in $[-\tau,0]\times\mathbb{R}$ and $u_{20}(x)\ge 0$ in $x\in\mathbb{R}$, then $u_i(t,x)\ge 0$ in $(0,+\infty)\times\mathbb{R},\ i=1,2$.*

Proof. We want to prove the non-negativity of $u_i(t,x)$ in each of the intervals $[n\tau,(n+1)\tau]$ one by one, $i=1,2,n=1,2,\ldots$. For $t\in[0,\tau]$, we have $-\tau\le t-\tau\le 0$, so $u_{10}(t-\tau,x)\ge 0$. Thus $g(u_{10}(t-\tau,x))\ge 0$ by the non-negativity of g. Therefore $u_2(t,x)$ satisfies the following differential inequality

$$\frac{\partial}{\partial t}u_2(t,x)+\beta u_2(t,x)=g(u_{10}(t-\tau,x))\ge 0,\quad t\in[0,\tau].$$

Applying the standard comparison principle for linear parabolic equations, we can obtain $u_2(t,x)\ge 0$ on $[0,\tau]$. By repeating this procedure to each of the intervals $[n\tau,(n+1)\tau],\ n=1,2,\ldots$, we have $u_2(t,x)\ge 0$ on $(0,+\infty)$. Similarly, by the non-negativity of $u_2(t,x)$ on $(0,+\infty)$, $u_1(t,x)$ also satisfies the following differential inequality

$$\frac{\partial}{\partial t}u_1(t,x)-\frac{\partial^2}{\partial x^2}u_1(t,x)+u_1(t,x)=u_2(t,x)\ge 0,\quad t\in[0,\tau].$$

Applying the standard comparison principle for linear parabolic equations again, we have $u_1(t,x)\ge 0$ on $[0,\tau]$. Repeating the same procedure to each of the intervals $[n\tau,(n+1)\tau],\ n=1,2,\ldots$, there holds $u_1(t,x)\ge 0$ on $(0,+\infty)$. This completes the proof. □

Next, we give the global existence and uniqueness result of solution (u_1,u_2) to the Cauchy problem (1.1) and (2.1).

Proposition 2.5 (*Global Existence and Uniqueness*)**.** *Assume that $u_{10}(s,x)\ge 0,u_{20}(x)\ge 0$ and are continuous for $s\in[-\tau,0],x\in\mathbb{R}$, respectively. For a given traveling wave solution $(\phi_1(x+ct),\phi_2(x+ct))$, if $u_{10}(s,x)-\phi_1(x+cs)\in C([-\tau,0];H^1(\mathbb{R}))$ and $u_{20}(x)-\phi_2(x)\in H^1(\mathbb{R})\subset C(\mathbb{R})$, then there exists a unique global solution $(u_1(t,x),u_2(t,x))$ of the Cauchy problem (1.1) and (2.1) such that $u_i(t,x)-\phi_i(x+ct)\in C([0,+\infty);H^1(\mathbb{R})),\ i=1,2$.*

Let $U_i(t,x)=u_i(t,x)-\phi_i(x+ct),\ i=1,2$, where $(\phi_1(x+ct),\phi_2(x+ct))$ is a given traveling wave solution of (1.1). Then the Cauchy problem (1.1) and (2.1) can be rewritten as

$$\frac{\partial U_1}{\partial t}(t,x)=\frac{\partial^2U_1(t,x)}{\partial x^2}-U_1(t,x)+U_2(t,x)\tag{2.3}$$

$$\frac{\partial U_2}{\partial t}(t,x)=-\beta U_2(t,x)+G(t-\tau,x),\quad (t,x)\in\mathbb{R}_+\times\mathbb{R},\tag{2.4}$$

with

$$\begin{cases}U_1(s,x)=U_{10}(s,x), & (s,x)\in[-\tau,0]\times\mathbb{R},\\ U_2(0,x)=U_{20}(x), & x\in\mathbb{R},\end{cases}\tag{2.5}$$

where $G(t-\tau,x)=g(U_1(t-\tau,x)+\phi_1(x+ct-c\tau))-g(\phi_1(x+ct-c\tau))$. Thus, Proposition 2.5 is equivalent to the following result.

Proposition 2.6. *Suppose the assumptions of Proposition 2.4 all hold, then there exists a unique global solution $(U_1(t,x), U_2(t,x))$ of the Cauchy problem (2.3)–(2.5) such that*

$$U_i(t,x) \in C\left([0,+\infty); H^1(\mathbb{R})\right), \quad i = 1, 2.$$

The proof of Proposition 2.5 depends on the following two results on the local existence, uniqueness, extension of solutions and the boundedness of solutions of (2.3)–(2.5).

Proposition 2.7 (*Local Existence and Uniqueness*). *For $U_{10}(s,x) \in C\left([-\tau, 0]; H^1(\mathbb{R})\right)$ and $U_{20}(x) \in H^1(\mathbb{R}) \subset C(\mathbb{R})$, $s \in [-\tau, 0]$, $x \in \mathbb{R}$, there exists $t_0 > 0$ such that problem (2.3)–(2.5) has a unique solution $(U_1(t,x), U_2(t,x)) \in C\left([0,t_0); H^1(\mathbb{R})\right)$. Furthermore, if $[0, T_0)$ is its maximal interval of existence and $(U_1(t,x), U_2(t,x)) \in C\left([0,T_0); H^1(\mathbb{R})\right)$, then either $T_0 = +\infty$ or $T_0 < +\infty$, and in the latter case $\lim_{t\to T_0^-} \|U_i(t,\cdot)\|_{H^1(\mathbb{R})} = +\infty$.*

It can also be proved by using the standard iteration method (see [50–52]). Thus the proof is omitted here.

Proposition 2.8 (*Boundedness*). *If $(U_1(t,x), U_2(t,x)) \in C\left([0,T]; H^1(\mathbb{R})\right)$ is a solution of (2.3)–(2.5) for $0 < T < \infty$. Then there exists positive constants C, independent of T, such that for $0 \le t < T$,*

$$\|U_i(t)\|_{H^1(\mathbb{R})} \le C\left(\|U_{10}(0)\|_{H^1}^2 + \|U_{20}(0)\|_{H^1}^2 + \int_{-\tau}^0 \|U_{10}(s)\|_{H^1}^2 ds\right) e^{\frac{\tilde{L}^2}{2\beta-1}t}, \quad i = 1, 2.$$

Proof. Multiplying (2.3) and (2.4) by $2U_1(t,x)$ and $2U_2(t,x)$, respectively, we have

$$(U_1^2)_t - 2(U_1U_{1x})_x + 2U_{1x}^2 + 2U_1^2 = 2U_1U_2, \tag{2.6}$$

$$(U_2^2)_t + 2\beta U_2^2 = 2G(t-\tau,x)U_2. \tag{2.7}$$

Integrating (2.6) and (2.7) over $[0,t] \times \mathbb{R}$, $t \in [0,T)$, respectively, we obtain

$$\begin{aligned}\|U_1(t)\|_{L^2}^2 + 2\int_0^t \|U_{1x}(s)\|_{L^2}^2 ds + 2\int_0^t \|U_1(s)\|_{L^2}^2 ds &= \|U_{10}(0)\|_{L^2}^2 + 2\int_0^t\int_{\mathbb{R}} U_1(s,x)U_2(s,x)dxds \\ &\le \|U_{10}(0)\|_{L^2}^2 + \int_0^t \|U_1(s)\|_{L^2}^2 ds + \int_0^t \|U_2(s)\|_{L^2}^2 ds,\end{aligned} \tag{2.8}$$

and

$$\|U_2(t)\|_{L^2}^2 + 2\beta\int_0^t \|U_2(s)\|_{L^2}^2 ds = \|U_{20}(0)\|_{L^2}^2 + 2\int_0^t\int_{\mathbb{R}} G(s-\tau,x)U_2(s,x)dxds. \tag{2.9}$$

Combining (2.8) and (2.9), we obtain

$$\begin{aligned}&\|U_1(t)\|_{L^2}^2 + \|U_2(t)\|_{L^2}^2 + 2\int_0^t \|U_{1x}(s)\|_{L^2}^2 ds + \int_0^t \|U_1(s)\|_{L^2}^2 ds + (2\beta-1)\int_0^t \|U_2(s)\|_{L^2}^2 ds \\ &\quad \le \|U_{10}(0)\|_{L^2}^2 + \|U_{20}(0)\|_{L^2}^2 + 2\int_0^t\int_{\mathbb{R}} G(s-\tau,x)U_2(s,x)dxds.\end{aligned} \tag{2.10}$$

By the mean-value theorem, there exists a function $\bar{U}_1(t,x)$, between $\phi_1(x+ct)$ and $\phi_1(x+ct) + U_1(t,x)$, such that

$$|G(s-\tau,x)| = |g'(\bar{U}_1)U_1(s-\tau,x)| \le \tilde{L}|U_1(s-\tau,x)|,$$

where $\tilde{L} = \max_{\bar{U}_1\in[0,+\infty)} |g'(\bar{U}_1)|$. Therefore, combining the Cauchy–Schwarz inequality, $|ab| \le \varepsilon a^2 + (\frac{1}{4\varepsilon})b^2$ for $\varepsilon > 0$, we have

$$\begin{aligned}2\left|\int_0^t\int_{\mathbb{R}} G(s-\tau,x)U_2(s,x)dxds\right| &\le 2\tilde{L}\int_0^t\int_{\mathbb{R}} |U_1(s-\tau,x)U_2(s,x)|dxds \\ &\le 2\tilde{L}\int_0^t\int_{\mathbb{R}}\left[\varepsilon U_2^2(s,x) + \frac{1}{4\varepsilon}U_1^2(s-\tau,x)\right]dxds \\ &= 2\tilde{L}\varepsilon\int_0^t \|U_2(s)\|_{L^2}^2 ds + \frac{\tilde{L}}{2\varepsilon}\int_{-\tau}^{t-\tau} \|U_1(s)\|_{L^2}^2 ds \\ &\le 2\tilde{L}\varepsilon\int_0^t \|U_2(s)\|_{L^2}^2 ds + \frac{\tilde{L}}{2\varepsilon}\int_{-\tau}^0 \|U_{10}(s)\|_{L^2}^2 ds + \frac{\tilde{L}}{2\varepsilon}\int_0^t \|U_1(s)\|_{L^2}^2 ds.\end{aligned} \tag{2.11}$$

Letting $\varepsilon = \frac{\beta-\frac{1}{2}}{\tilde{L}}$ in (2.11), we obtain

$$2\int_0^t \int_{\mathbb{R}} G(s-\tau, x)U_2(s,x)dxds \le (2\beta-1)\int_0^t \|U_2(s)\|_{L^2}^2 ds + \frac{\tilde{L}^2}{2\beta-1}\left\{\int_{-\tau}^0 \|U_{10}(s)\|_{L^2}^2 ds + \int_0^t \|U_1(s)\|_{L^2}^2 ds\right\}.$$

Substituting the above inequality into (2.10), we have

$$\|U_1(t)\|_{L^2}^2 + \|U_2(t)\|_{L^2}^2 + 2\int_0^t \|U_{1x}(s)\|_{L^2}^2 ds \le \|U_{10}(0)\|_{L^2}^2 + \|U_{20}(0)\|_{L^2}^2 + \frac{\tilde{L}^2}{2\beta-1}\int_{-\tau}^0 \|U_{10}(s)\|_{L^2}^2 ds + \frac{\tilde{L}^2}{2\beta-1}\int_0^t \|U_1(s)\|_{L^2}^2 ds. \tag{2.12}$$

Applying Gronwall's inequality to (2.12), we have

$$\|U_1(t)\|_{L^2}^2 + \|U_2(t)\|_{L^2}^2 \le \left(\|U_{10}(0)\|_{L^2}^2 + \|U_{20}(0)\|_{L^2}^2 + \frac{\tilde{L}^2}{2\beta-1}\int_{-\tau}^0 \|U_{10}(s)\|_{L^2}^2 ds\right) e^{\frac{\tilde{L}^2}{2\beta-1}t}$$

for $t \in [0, T)$, i.e.

$$\|U_i(t)\|_{L^2}^2 \le \left(\|U_{10}(0)\|_{L^2}^2 + \|U_{20}(0)\|_{L^2}^2 + \frac{\tilde{L}^2}{2\beta-1}\int_{-\tau}^0 \|U_{10}(s)\|_{L^2}^2 ds\right) e^{\frac{\tilde{L}^2}{2\beta-1}t}, \tag{2.13}$$

for $t \in [0, T)$ and $i = 1, 2$. Similarly, we can prove that

$$\|U_{ix}(t)\|_{L^2}^2 \le C\left(\|U_{10}(0)\|_{L^2}^2 + \|U_{20}(0)\|_{L^2}^2 + \frac{\tilde{L}^2}{2\beta-1}\int_{-\tau}^0 \|U_{10}(s)\|_{L^2}^2 ds\right) e^{\frac{\tilde{L}^2}{2\beta-1}t}, \tag{2.14}$$

for some positive $C > 0$, $t \in [0, T)$ and $i = 1, 2$.

Therefore, relations (2.13) and (2.14) lead to

$$\|U_i(t)\|_{H^1(\mathbb{R})}^2 \le C\left(\|U_{10}(0)\|_{H^1}^2 + \|U_{20}(0)\|_{H^1}^2 + \int_{-\tau}^0 \|U_{10}(s)\|_{H^1}^2 ds\right) e^{\frac{\tilde{L}^2}{2\beta-1}t}, \quad 0 \le t < T,$$

$i = 1, 2$, for some positive $C > 0$. □

Proposition 2.6 now follows from Propositions 2.7 and 2.8, and Proposition 2.5 follows immediately from Propositions 2.4 and 2.6.

Define a weight function as

$$w(\xi) = \begin{cases} e^{-\gamma(\xi-\xi_*)}, & \text{for } \xi < \xi_*, \\ 1, & \text{for } \xi \ge \xi_*, \end{cases} \tag{2.15}$$

where $\gamma = \frac{c_*}{2}$. Next, we are going to state our main result regarding the exponential asymptotic stability of the traveling wave solutions of system (1.1).

Theorem 2.9 (*Stability*). *Suppose that $g \in C^3([0, K^+], \mathbb{R})$, $|g'(K)| < \min\{1, \frac{2\beta-1}{2}\}$ and $\beta > \frac{1}{2}$. For a given traveling wave solution $\mathbf{\Phi}(x+ct) = (\phi_1(x+ct), \phi_2(x+ct))$ of (1.1) with the speed c satisfying*

$$c > \max\left\{\frac{2(L+L_1+1-2\beta)}{c_*}, c_*\right\}. \tag{2.16}$$

If the initial perturbation is

$$\begin{cases} u_{10}(s,x) - \phi_1(x+cs) \in C\left([-\tau, 0]; H_w^1(\mathbb{R})\right), \\ u_{20}(x) - \phi_2(x) \in H_w^1(\mathbb{R}) \subset C(\mathbb{R}), \end{cases}$$

where $w(x)$ is the weighted function given in (2.15), then there exist positive constants $\delta_0 = \delta_0(\beta, \tau, g, c)$ and $\mu = \mu(\beta, \tau, g, c)$ such that, when $\sup_{s\in[-\tau,0]}\left(\|u_{10}(s,\cdot) - \phi_1(\cdot+cs)\|_{H_w^1(\mathbb{R})}\right) + \|u_{20}(\cdot) - \phi_2(\cdot)\|_{H_w^1(\mathbb{R})} \le \delta_0$, the unique solution $(u_1(t,x), u_2(t,x))$

of the Cauchy problem (1.1) and (2.1) exists globally, and satisfies

$$u_i(t,x)-\phi_i(x+ct)\in C\left([0,\infty];H_w^1(\mathbb{R})\right)\cap L^2\left([0,+\infty);H_w^2(\mathbb{R})\right),\quad i=1,2$$

and

$$\sup_{x\in\mathbb{R}}|u_i(t,x)-\phi_i(x+ct)|\le Ce^{-\mu t},\quad t\ge 0,\ i=1,2.$$

3. Proof of main result

This section is devoted to the proof of the stability result, i.e., Theorem 2.9, by means of the weighted energy method.

Let $(u_1(t,x),u_2(t,x))$ be the solution of the Cauchy problem (1.1), (1.2) and (2.1), and $(\phi_1(x+ct),\phi_2(x+ct))$ be a given traveling wave solution of (1.1). Let $\xi:=x+ct$ and

$$\begin{cases}V_i(t,\xi)=u_i(t,x)-\phi_i(\xi),\quad i=1,2,\\ V_{10}(s,\xi)=u_{10}(s,x)-\phi_1(x+cs),\quad (s,x)\in[-\tau,0]\times\mathbb{R},\\ V_{20}(\xi)=u_{20}(x)-\phi_2(x),\quad x\in\mathbb{R}.\end{cases}\tag{3.1}$$

Then the original problem (1.1) and (2.1) can be reformulated as

$$V_{1t}(t,\xi)+cV_{1\xi}(t,\xi)=V_{1\xi\xi}(t,\xi)-V_1(t,\xi)+V_2(t,\xi)\tag{3.2}$$

$$V_{2t}(t,\xi)+cV_{2\xi}(t,\xi)+\beta V_2(t,\xi)-g'\left(\phi_1(\xi-c\tau)\right)V_1(t-\tau,\xi-c\tau)=Q(t-\tau,\xi-c\tau)\tag{3.3}$$

with the initial conditions

$$\begin{cases}V_1(s,\xi)=V_{10}(s,\xi),\quad (s,\xi)\in[-\tau,0]\times\mathbb{R},\\ V_2(0,\xi)=V_{20}(\xi),\quad \xi\in\mathbb{R},\end{cases}\tag{3.4}$$

where

$$Q(t-\tau,\xi-c\tau)=g\left(\phi_1(\xi-c\tau)+V_1(t-\tau,\xi-c\tau)\right)-g\left(\phi_1(\xi-c\tau)\right)-g'\left(\phi_1(\xi-c\tau)\right)V_1(t-\tau,\xi-c\tau).$$

Therefore, Theorem 2.9 is equivalent to the following result.

Theorem 3.1. *Suppose that $g\in C^3\left([0,K^+],\mathbb{R}\right)$, $\beta>\frac{1}{2}$, $|g'(K)|<\min\{1,\frac{2\beta-1}{2}\}$. For a given traveling wave solution $(\phi_1(\xi),\phi_2(\xi))$ of (1.1) with the speed c satisfying (2.16). If*

$$\begin{cases}V_{10}(s,x)-\phi_1(x+cs)\in C\left([-\tau,0];H_w^1(\mathbb{R})\right),\\ V_{20}(x)-\phi_2(x)\in H_w^1(\mathbb{R})\subset C(\mathbb{R}),\end{cases}$$

where $w(x)$ is the weighted function given in (2.15), then there exist positive constants $\delta_0=\delta_0(\beta,\tau,g,c)$ and $\mu=\mu(\beta,\tau,g,c)$ such that, when $\sup_{s\in[-\tau,0]}\left(\|V_{10}(s,\cdot)\|_{H_w^1(\mathbb{R})}\right)+\|V_{20}(\cdot)\|_{H_w^1(\mathbb{R})}\le\delta_0$, the unique solution $\mathbf{V}(t,\xi)=(V_1(t,\xi),V_2(t,\xi))$ of the Cauchy problem (3.2)–(3.4) exists globally, and satisfies

$$V_i(t,\xi)\in C\left([0,\infty];H_w^1(\mathbb{R})\right)\cap L^2\left([0,+\infty);H_w^2(\mathbb{R})\right),\quad i=1,2$$

and

$$\sup_{x\in\mathbb{R}}|V_i(t,\xi)|\le Ce^{-\mu t},\quad t\ge 0,\ i=1,2.\tag{3.5}$$

Let

$$\begin{aligned}X(r-\tau,r+T)=\Big\{&(V_1,V_2)|V_2\in C\big([r,r+T];H_w^1(\mathbb{R})\big)\cap L^2\big([r,r+T];H_w^2(\mathbb{R})\big),\\ &V_1\in C\big([r-\tau,r+T];H_w^1(\mathbb{R})\big)\cap L^2\big([r-\tau,r+T];H_w^2(\mathbb{R})\big)\Big\}\end{aligned}$$

and

$$M_r(T):=\max\left\{\sup_{t\in[r-\tau,r]}\|V_1(t)\|_{H_w^1},\ \sup_{t\in[r,r+T]}\left(\|V_1(t)\|_{H_w^1}^2+\|V_2(t)\|_{H_w^1}^2\right)^{\frac{1}{2}}\right\},$$

where $\tau\ge 0$ and $T>0$. When $r=0$, we denote $M(T)=M_0(T)$. For simplicity, in what follows, we denote $V_i(t)=V_i(t,\cdot)$.

With the help of the continuity argument, the proof of Theorem 3.1 depends on the following two results about local estimate of solutions and a prior estimate.

Proposition 3.2 (*Local Estimate*). *Consider the Cauchy problem with the initial time* $r \geq 0$,

$$\begin{cases} V_{1t}(t,\xi) + cV_{1\xi}(t,\xi) = V_{1\xi\xi}(t,\xi) - V_1(t,\xi) + V_2(t,\xi), \\ V_{2t}(t,\xi) + cV_{2\xi}(t,\xi) + \beta V_2(t,\xi) \\ \qquad - g'(\phi_1(\xi - c\tau))V_1(t-\tau, \xi - c\tau) = Q(t-\tau, \xi - c\tau), \quad (t,\xi) \in [r, +\infty) \times \mathbb{R} \\ V_{10}(s,\xi) = u_{10}(s,\xi) - \phi_1(\xi - cs) := V_{1r}(s,\xi), \quad (s,\xi) \in [r-\tau, r] \times \mathbb{R}, \\ V_{20}(\xi) = u_{20}(\xi) - \phi_2(\xi) := V_{2r}(0,\xi), \quad \xi \in \mathbb{R}. \end{cases} \tag{3.6}$$

If $V_{1r}(s,\xi) \in H^1_w(\mathbb{R}),\ s \in [-\tau, 0], V_{2r}(0,\xi) \in H^1_w(\mathbb{R})$ *and* $M_r(0) \leq \delta_1$ *for a given constant* $\delta_1 > 0$, *then there exists* $t_0 = t_0(\delta_1) > 0$ *such that* $\mathbf{V}(t,\xi) \in X(r-\tau, r+t_0)$ *and* $M_r(t_0) \leq \sqrt{2(1+\tau)}M_r(0)$.

The proof can also be derived from an elementary energy method and a standard method, so we omit it here.

Proposition 3.3 (*A Prior Estimate*). *Let* $\mathbf{V}(t,\xi) = (V_1(t,\xi), V_2(t,\xi))$ *be a local solution of the Cauchy problem* (3.2)–(3.4), *where* $\mathbf{V}(t,\xi) \in X(-\tau, T),\ i = 1, 2$. *Then there exist positive constants* μ, δ_2 *and* $C_7 > 0$ *independent of* $T > 0$ *such that, if* $M(T) \leq \delta_2$, *then for* $0 \leq t \leq T,\ i = 1, 2$,

$$e^{2\mu t}\left(\sum_{i=1}^{2} \|V_i(t)\|^2_{H^1_w}\right) + \int_0^t e^{2\mu s}\|V_1(s)\|^2_{H^1_w} ds \leq C_7\left(\|V_{10}(0)\|^2_{H^1_w} + \|V_{20}(0)\|^2_{H^1_w} + \int_{-\tau}^0 \|V_{10}(s)\|^2_{H^1_w} ds\right) \tag{3.7}$$

and

$$\sum_{i=1}^{2} \|V_i(t)\|^2_{H^1_w} \leq C_7\left(\|V_{10}(0)\|^2_{H^1_w} + \|V_{20}(0)\|^2_{H^1_w} + \int_{-\tau}^0 \|V_{10}(s)\|^2_{H^1_w} ds\right) e^{-2\mu t}. \tag{3.8}$$

In order to obtain the prior estimate (Proposition 3.3), we make an extension for the function g as follows:

$$\bar{g}(u) = \begin{cases} g(0) + g'(0)u + \dfrac{g''(0)}{2!}u^2 + \dfrac{g'''(0)}{3!}u^3, \quad u \in [-\sigma K^+, 0) \\ g(u), \quad u \in [0, K^+] \\ g(K^+) + g'(K^+)(u - K^+) + \dfrac{g''(K^+)}{2!}(u-K^+)^2 + \dfrac{g'''(K^+)}{3!}(u-K^+)^3, \quad u \in (K^+, (\sigma+1)K^+], \end{cases}$$

where $\sigma \geq 1$ is any fixed constant and $K^+ > 0$ is defined in (C_2).

Define

$$B^+_{\mu,w}(\xi) = -c\frac{w'(\xi)}{w(\xi)} + 1 - 2\mu - \left(\frac{w'(\xi)}{w(\xi)}\right)^2 - e^{2\mu\tau}\frac{w(\xi + c\tau)}{w(\xi)}|g'(\phi_1(\xi))|, \tag{3.9}$$

$$B^-_{\mu,w}(\xi) = -c\frac{w'(\xi)}{w(\xi)} + 2\beta - 2\mu - 1 - |g'(\phi_1(\xi - c\tau))|, \tag{3.10}$$

$$\bar{B}^+_{\mu,w}(\xi) = B^+_{\mu,w}(\xi) \tag{3.11}$$

$$\bar{B}^-_{\mu,w}(\xi) = B^-_{\mu,w}(\xi) - |g''(\phi_1(\xi - c\tau))\phi_1'(\xi - c\tau)| \tag{3.12}$$

and

$$C^+_1(\mu) = \frac{1}{4}cc_* + 1 - 2\mu - Le^{2\mu\tau}, \qquad C^+_2(\mu) = \frac{1 - |g'(K)|}{2} - 2\mu - L\big(e^{2\mu\tau} - 1\big),$$
$$C^-_1(\mu) = \frac{1}{2}cc_* + 2\beta - 1 - L - 2\mu, \qquad C^-_2(\mu) = \frac{2\beta - 1}{2} - |g'(K)| - 2\mu,$$
$$C^+_3(\mu) = C^+_1(\mu), \qquad C^+_4(\mu) = C^+_2(\mu), \qquad C^-_3(\mu) = C^-_1(\mu) - L_1, \qquad C^-_4(\mu) = C^-_2(\mu).$$

In order to prove Proposition 3.3, we need to prove the following result. For the sake of convenience, we denote $\bar{g}$ by g in the following.

Lemma 3.4 (*Key Inequality*). *Let* $w(\xi)$ *be the weight function given in* (2.15), *if* (2.16) *holds and* $|g'(K)| < \min\{1, \frac{2\beta-1}{2}\}$, *then for all* $\xi \in \mathbb{R}$ *and* $0 < \mu < \mu_0 := \min\{\mu_i',\ i = 1, 2, \ldots, 8\}$,

$$B^{\pm}_{\mu,w}(\xi) > C^{\pm}_0(\mu) = \min\{C^{\pm}_1(\mu), C^{\pm}_2(\mu)\} > 0, \tag{3.13}$$

and

$$\bar{B}^{\pm}_{\mu,w}(\xi) > \bar{C}^{\pm}_0(\mu) = \min\{C^{\pm}_3(\mu), C^{\pm}_4(\mu)\} > 0, \tag{3.14}$$

where $\mu'_i > 0$ $(i = 1, 2, \dots, 8)$ *are the unique solution to the equation* $C^{\pm}_i(\mu) = 0$ $(i = 1, 2)$ *and* $\bar{C}^{\pm}_i(\mu) = 0$ $(i = 3, 4)$*, respectively.*

Proof. We distinguish among two cases: (we only prove (3.13), and the proof of (3.14) is similar). First we prove $B^{+}_{\mu,w}(\xi) > C^{+}_0(\mu) > 0$ holds.

Case 1: $\xi < \xi_*$. From (2.15), we have $w(\xi) = e^{-\gamma(\xi-\xi_*)}$. Note also that $w(\cdot)$ is non-increasing. Thus we obtain

$$\begin{aligned} B^{+}_{\mu,w}(\xi) &= -c(-\gamma) + 1 - 2\mu - (-\gamma)^2 - e^{2\mu\tau}\frac{w(\xi+c\tau)}{w(\xi)}|g'(\phi_1(\xi))| \\ &\geq \gamma c + 1 - \gamma^2 - 2\mu - Le^{2\mu\tau} \\ &> \frac{1}{4}cc_* + 1 - 2\mu - Le^{2\mu\tau} = C^{+}_1(\mu) > 0. \end{aligned}$$

Case 2: $\xi \geq \xi_*$. In this case, $w(\xi) = w(\xi + c\tau) = 1$, $\frac{w'(\xi)}{w(\xi)} = 0$. Thus, we have

$$\begin{aligned} B^{+}_{\mu,w}(\xi) &= 1 - 2\mu - e^{2\mu\tau}|g'(\phi_1(\xi))| \\ &= (1 - |g'(\phi_1(\xi))|) - 2\mu - \big(e^{2\mu\tau} - 1\big)|g'(\phi_1(\xi))| \\ &> (1 - |g'(K)| - \epsilon) - 2\mu - L\big(e^{2\mu\tau} - 1\big) \\ &\geq \frac{1 - |g'(K)|}{2} - 2\mu - L\big(e^{2\mu\tau} - 1\big) = C^{+}_2(\mu) > 0. \end{aligned}$$

Let $C^{+}_0(\mu) = \min\{C^{+}_1(\mu), C^{+}_2(\mu)\}$, then $B^{+}_{\mu,w}(\xi) > C^{+}_0(\mu) > 0$ holds.

Next, we prove that $B^{-}_{\mu,w}(\xi) > C^{-}_0(\mu) > 0$ holds.

Case 1: $\xi < \xi_*$. Thus,

$$\begin{aligned} B^{-}_{\mu,w}(\xi) &= -c(-\gamma) + 2\beta - 2\mu - 1 - |g'(\phi_1(\xi - c\tau))| \\ &\geq \frac{cc_*}{2} + 2\beta - 1 - 2\mu - L = C^{-}_1(\mu) > 0. \end{aligned}$$

Case 2: $\xi \geq \xi_*$. From (2.15), we have $w(\xi) = 1$, $\frac{w'(\xi)}{w(\xi)} = 0$. By Lemma 2.3, we get

$$\begin{aligned} B^{-}_{\mu,w}(\xi) &= 2\beta - 2\mu - 1 - |g'(\phi_1(\xi - c\tau))| \\ &\geq 2\beta - 1 - |g'(K)| - \frac{\epsilon}{4} - 2\mu \\ &\geq 2\beta - 1 - |g'(K)| - \frac{2\beta - 1}{4} - 2\mu \\ &> \frac{2\beta - 1}{2} - |g'(K)| - 2\mu = C^{-}_2(\mu) > 0. \end{aligned}$$

Let $C^{-}_0(\mu) = \min\{C^{-}_1(\mu), C^{-}_2(\mu)\}$, then $B^{-}_{\mu,w}(\xi) > C^{-}_0(\mu) > 0$ holds too.

The proof of (3.14) is similar, so we omit it here. □

Next, we begin to prove Proposition 3.3.

Proof. Multiplying (3.2) and (3.3) by $e^{2\mu t}w(\xi)V_1(t,\xi)$ and $e^{2\mu t}w(\xi)V_2(t,\xi)$, respectively, we obtain

$$\begin{aligned} &\left\{\frac{1}{2}e^{2\mu t}wV_1^2\right\}_t + \left\{\frac{1}{2}cwV_1^2 - wV_1V_{1\xi}\right\}_\xi e^{2\mu t} + e^{2\mu t}wV_{1\xi}^2 + e^{2\mu t}w'V_1V_{1\xi} \\ &\quad + \left\{-\frac{c}{2}\left(\frac{w'}{w}\right) + 1 - \mu\right\}e^{2\mu t}wV_1^2 = e^{2\mu t}wV_1V_2, \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} &\left\{\frac{1}{2}e^{2\mu t}wV_2^2\right\}_t + \left\{\frac{1}{2}cwV_2^2\right\}_\xi e^{2\mu t} + \left\{-\frac{c}{2}\left(\frac{w'}{w}\right) + \beta - \mu\right\}e^{2\mu t}wV_2^2 \\ &\quad = e^{2\mu t}wg'\left(\phi_1(\xi - c\tau)\right)V_1(t - \tau, \xi - c\tau)V_2 + e^{2\mu t}wV_2Q(t - \tau, \xi - c\tau). \end{aligned} \tag{3.16}$$

For (3.15), by the Cauchy–Schwarz inequality $xy \leq \frac{x^2}{2} + \frac{y^2}{2}$, we have

$$|e^{2\mu t} w' V_{1\xi} V_1| = e^{2\mu t} w |V_{1\xi} \frac{w'}{w} V_1| \leq \frac{e^{2\mu t} w V_{1\xi}^2}{2} + \frac{1}{2} e^{2\mu t} \left(\frac{w'}{w}\right)^2 w V_1^2,$$

then (3.15) is reduced to

$$\begin{aligned} &\left\{\frac{1}{2} e^{2\mu t} w V_1^2\right\}_t + \left\{\frac{1}{2} c w V_1^2 - w V_1 V_{1\xi}\right\}_\xi e^{2\mu t} + \frac{1}{2} e^{2\mu t} w V_{1\xi}^2 \\ &\quad + \left\{-\frac{c}{2}\left(\frac{w'}{w}\right) + 1 - \mu - \frac{1}{2}\left(\frac{w'}{w}\right)^2\right\} e^{2\mu t} w V_1^2 \leq e^{2\mu t} w V_1 V_2. \end{aligned} \tag{3.17}$$

Integrating (3.17) over $R \times [0, t]$ with respect to ξ and t, we further have

$$\begin{aligned} &e^{2\mu t} \|V_1(t)\|_{L_w^2}^2 + \int_0^t e^{2\mu s} \|V_{1\xi}(s)\|_{L_w^2}^2 ds + \int_0^t \int_R \left\{-c\left(\frac{w'}{w}\right) + 2 - 2\mu - \left(\frac{w'}{w}\right)^2\right\} e^{2\mu s} w(\xi) V_1^2(s, \xi) d\xi ds \\ &\quad \leq \|V_{10}(0)\|_{L_w^2}^2 + 2\int_0^t \int_R e^{2\mu s} w(\xi) V_1(s, \xi) V_2(s, \xi) d\xi ds. \end{aligned} \tag{3.18}$$

Again, using the Cauchy–Schwarz inequality we have

$$|2e^{2\mu s} w(\xi) V_1(s, \xi) V_2(s, \xi)| \leq e^{2\mu s} w(\xi)\left[V_1^2(s, \xi) + V_2^2(s, \xi)\right].$$

Thus, (3.18) is reduced to

$$\begin{aligned} &e^{2\mu t} \|V_1(t)\|_{L_w^2}^2 + \int_0^t e^{2\mu s} \|V_{1\xi}(s)\|_{L_w^2}^2 ds + \int_0^t \int_R \left\{-c\left(\frac{w'}{w}\right) + 1 - 2\mu - \left(\frac{w'}{w}\right)^2\right\} e^{2\mu s} w(\xi) V_1^2(s, \xi) d\xi ds \\ &\quad \leq \|V_{10}(0)\|_{L_w^2}^2 + \int_0^t \int_R e^{2\mu s} w(\xi) V_1^2(s, \xi) d\xi ds + \int_0^t \int_R e^{2\mu s} w(\xi) V_2^2(s, \xi) d\xi ds. \end{aligned} \tag{3.19}$$

On the other hand, integrating (3.16) over $R \times [0, t]$ with respect to ξ and t, we further have

$$\begin{aligned} &e^{2\mu t} \|V_2(t)\|_{L_w^2}^2 + \int_0^t \int_R \left\{-c\left(\frac{w'}{w}\right) + 2\beta - 2\mu\right\} e^{2\mu s} w(\xi) V_2^2(s, \xi) d\xi ds \\ &\quad - 2\int_0^t \int_R e^{2\mu s} w(\xi) g'\left(\phi_1(\xi - c\tau)\right) V_1(s - \tau, \xi - c\tau) V_2(s, \xi) d\xi ds \\ &\quad \leq \|V_{20}(0)\|_{L_w^2}^2 + 2\int_0^t \int_R e^{2\mu s} w(\xi) V_2(s, \xi) Q(s - \tau, \xi - c\tau) d\xi ds. \end{aligned} \tag{3.20}$$

Using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} &|2e^{2\mu s} w(\xi) g'(\phi_1(\xi - c\tau)) V_1(s - \tau, \xi - c\tau) V_2(s, \xi)| \\ &\quad \leq e^{2\mu s} w(\xi) g'\left(\phi_1(\xi - c\tau)\right)\left[V_1^2(s - \tau, \xi - c\tau) + V_2^2(s, \xi)\right]. \end{aligned}$$

Thus, the third term on the left-hand-side of (3.20) is reduced to

$$\begin{aligned} &\left|2\int_0^t \int_R e^{2\mu s} w(\xi) g'\left(\phi_1(\xi - c\tau)\right) V_1(s - \tau, \xi - c\tau) V_2(s, \xi) d\xi ds\right| \\ &\quad \leq \int_0^t \int_R e^{2\mu s} w(\xi) |g'(\phi_1(\xi - c\tau))| V_1^2(s - \tau, \xi - c\tau) d\xi ds \\ &\qquad + \int_0^t \int_R e^{2\mu s} w(\xi) |g'(\phi_1(\xi - c\tau))| V_2^2(s, \xi) d\xi ds \\ &\quad = e^{2\mu\tau} \int_{-\tau}^{t-\tau} \int_R e^{2\mu s} w(\xi + c\tau) |g'(\phi_1(\xi))| V_1^2(s, \xi) d\xi ds + \int_0^t \int_R e^{2\mu s} w(\xi) |g'(\phi_1(\xi - c\tau))| V_2^2(s, \xi) d\xi ds \\ &\quad \leq e^{2\mu\tau} \int_{-\tau}^0 \int_R e^{2\mu s} w(\xi + c\tau) |g'(\phi_1(\xi))| V_{10}^2(s, \xi) d\xi ds \\ &\qquad + e^{2\mu\tau} \int_0^t \int_R e^{2\mu s} w(\xi + c\tau) |g'(\phi_1(\xi))| V_1^2(s, \xi) d\xi ds + \int_0^t \int_R e^{2\mu s} w(\xi) |g'(\phi_1(\xi - c\tau))| V_2^2(s, \xi) d\xi ds. \end{aligned}$$

Therefore, (3.20) is reduced to

$$
\begin{aligned}
&e^{2\mu t}\|V_2(t)\|^2_{L^2_w} + \int_0^t \int_R \left\{-c\left(\frac{w'}{w}\right) + 2\beta - 2\mu\right\} e^{2\mu s} w(\xi) V_2^2(s,\xi)d\xi ds \\
&\quad - e^{2\mu\tau}\int_0^t \int_R e^{2\mu s} w(\xi + c\tau)|g'(\phi_1(\xi))| V_1^2(s,\xi)d\xi ds \\
&\quad - \int_0^t \int_R e^{2\mu s} w(\xi)|g'(\phi_1(\xi - c\tau))| V_2^2(s,\xi)d\xi ds \\
&= e^{2\mu\tau}\int_{-\tau}^0 \int_R e^{2\mu s} w(\xi)|g'(\phi_1(\xi))| \frac{w(\xi + c\tau)}{w(\xi)} V_{10}^2(s,\xi)d\xi ds + \|V_{20}(0)\|^2_{L^2_w} \\
&\quad + 2\int_0^t \int_R e^{2\mu s} w(\xi) V_2(s,\xi) Q(s-\tau, \xi - c\tau)d\xi ds.
\end{aligned}
\tag{3.21}
$$

Combining (3.19) and (3.21), we get

$$
\begin{aligned}
&e^{2\mu t}\left(\|V_1(t)\|^2_{L^2_w} + \|V_2(t)\|^2_{L^2_w}\right) + \int_0^t e^{2\mu s}\|V_{1\xi}(s)\|^2_{L^2_w} ds \\
&\quad + \int_0^t \int_R \left(\left\{-c\left(\frac{w'}{w}\right) + 1 - 2\mu - \left(\frac{w'}{w}\right)^2\right\} - e^{2\mu\tau}|g'(\phi_1(\xi))| \frac{w(\xi + c\tau)}{w(\xi)}\right) e^{2\mu s} w(\xi) V_1^2(s,\xi)d\xi ds \\
&\quad + \int_0^t \int_R \left(\left\{-c\left(\frac{w'}{w}\right) + 2\beta - 2\mu - 1\right\} - |g'(\phi_1(\xi - c\tau))|\right) e^{2\mu s} w(\xi) V_2^2(s,\xi)d\xi ds \\
&\le \|V_{10}(0)\|^2_{L^2_w} + \|V_{20}(0)\|^2_{L^2_w} + e^{2\mu\tau}\int_{-\tau}^0 \int_R e^{2\mu s} w(\xi)|g'(\phi_1(\xi))| \frac{w(\xi + c\tau)}{w(\xi)} V_{10}^2(s,\xi)d\xi ds \\
&\quad + 2\int_0^t \int_R e^{2\mu s} w(\xi) V_2(s,\xi) Q(s-\tau, \xi - c\tau)d\xi ds \\
&\le \|V_{10}(0)\|^2_{L^2_w} + \|V_{20}(0)\|^2_{L^2_w} + Le^{2\mu\tau}\int_{-\tau}^0 \int_R e^{2\mu s} w(\xi) V_{10}^2(s,\xi)d\xi ds \\
&\quad + 2\int_0^t \int_R e^{2\mu s} w(\xi) V_2(s,\xi) Q(s-\tau, \xi - c\tau)d\xi ds
\end{aligned}
$$

i.e.

$$
\begin{aligned}
&e^{2\mu t}\left(\|V_1(t)\|^2_{L^2_w} + \|V_2(t)\|^2_{L^2_w}\right) + \int_0^t e^{2\mu s}\|V_{1\xi}(s)\|^2_{L^2_w} ds \\
&\quad + \int_0^t \int_R e^{2\mu s} B^+_{\mu,w}(\xi) w(\xi) V_1^2(s,\xi)d\xi ds + \int_0^t \int_R e^{2\mu s} B^-_{\mu,w}(\xi) w(\xi) V_2^2(s,\xi)d\xi ds \\
&\le \|V_{10}(0)\|^2_{L^2_w} + \|V_{20}(0)\|^2_{L^2_w} + Le^{2\mu\tau}\int_{-\tau}^0 \int_R e^{2\mu s} w(\xi) V_{10}^2(s,\xi)d\xi ds \\
&\quad + 2\int_0^t \int_R e^{2\mu s} w(\xi) V_2(s,\xi) Q(s-\tau, \xi - c\tau)d\xi ds,
\end{aligned}
\tag{3.22}
$$

where $B^+_{\mu,w}(\xi)$ and $B^-_{\mu,w}(\xi)$ are defined in (3.9) and (3.10), respectively. Therefore, by (3.13) in Lemma 3.4, we have

$$
\begin{aligned}
&e^{2\mu t}\left(\|V_1(t)\|^2_{L^2_w} + \|V_2(t)\|^2_{L^2_w}\right) + \int_0^t e^{2\mu s}\|V_{1\xi}(s)\|^2_{L^2_w} ds \\
&\quad + C_0^+(\mu)\int_0^t e^{2\mu s}\| \|V_1(s)\|^2_{L^2_w} ds + C_0^-(\mu)\int_0^t e^{2\mu s}\| \|V_2(s)\|^2_{L^2_w} ds \\
&\le \|V_{10}(0)\|^2_{L^2_w} + \|V_{20}(0)\|^2_{L^2_w} + C_1\int_{-\tau}^0 e^{2\mu s}\|V_{10}(s)\|^2_{L^2_w} ds \\
&\quad + 2\int_0^t \int_R e^{2\mu s} w(\xi) V_2(s,\xi) Q(s-\tau, \xi - c\tau)d\xi ds,
\end{aligned}
\tag{3.23}
$$

where $C_1 = Le^{2\mu\tau} > 0$.

In addition, when $M(T) < K^+$, by (3.24), we have

$$|V_i(t,\xi)| \le \sup_{\xi\in\mathbb{R}} |V_i(t,\xi)| \le C_2\|V_i(t,\xi)\|_{H^1} \le C_2\|V_i(t,\xi)\|_{H^1_w} \le C_2M(T) < \sigma K^+$$

for all $t \in [-\tau, T]$. Thus, for the nonlinearity $Q(t-\tau, \xi - c\tau)$, using Taylor's formula, we have

$$|Q(t-\tau,\xi-c\tau)| = \frac{|g''(\eta)|}{2!}V_1^2(t-\tau,\xi-c\tau) \le L_2V_1^2(t-\tau,\xi-c\tau),$$

where $\eta = \phi_1 + \theta V_1 \in [-\sigma K^+, (\sigma+1)K^+]$, $\theta \in (0, 1)$. By the standard Sobolev embedding inequality $H^1(\mathbb{R}) \hookrightarrow C(\mathbb{R})$ and the modified embedding inequality $H^1_w(\mathbb{R}) \hookrightarrow H^1(\mathbb{R})$ for $w(\xi) > 0$ defined in (2.15), we obtain

$$|V_i(t,\xi)| \le \sup_{\xi\in\mathbb{R}} |V_i(t,\xi)| \le C_2\|V_i(t,\xi)\|_{H^1} \le C_2\|V_i(t,\xi)\|_{H^1_w} \le C_2M(t), \tag{3.24}$$

(because $\sum_{i=1}^2 \|V_i(t,\xi)\|^2_{H^1_w} \le M^2(t)$ by the definition of $M(t)$) where $C_2 > 0$ is the embedding constant. Therefore

$$\begin{aligned}
&2\left|\int_0^t\int_R e^{2\mu s}w(\xi)V_2(s,\xi)Q(s-\tau,\xi-c\tau)d\xi ds\right| \\
&\quad\le 2\int_0^t\int_R e^{2\mu s}w(\xi)|V_2(s,\xi)|\,|Q(s-\tau,\xi-c\tau)|d\xi ds \\
&\quad\le 2C_2L_2M(t)\int_0^t\int_R e^{2\mu s}w(\xi)V_1^2(s-\tau,\xi-c\tau)d\xi ds \\
&\quad= 2C_2L_2M(t)\int_{-\tau}^{t-\tau}\int_R w(\xi)e^{2\mu(s+\tau)}\frac{w(\xi+c\tau)}{w(\xi)}V_1^2(s,\xi)d\xi ds \\
&\quad\le C_3M(t)\left\{\int_{-\tau}^0 e^{2\mu s}\|V_{10}(s)\|^2_{L^2_w}ds + \int_0^t e^{2\mu s}\|V_1(s)\|^2_{L^2_w}ds\right\},
\end{aligned}\tag{3.25}$$

for some positive constant $C_3 > 0$. Substituting (3.25) into (3.23), we have

$$\begin{aligned}
&e^{2\mu t}\left(\|V_1(t)\|^2_{L^2_w} + \|V_2(t)\|^2_{L^2_w}\right) + \int_0^t e^{2\mu s}\|V_{1\xi}(s)\|^2_{L^2_w}ds \\
&\quad+\left[C_0^+(\mu) - C_3M(t)\right]\int_0^t e^{2\mu s}\|\,\|V_1(s)\,\|^2_{L^2_w}ds + C_0^-(\mu)\int_0^t e^{2\mu s}\|\,\|V_2(s)\|^2_{L^2_w}ds \\
&\quad\le \|V_{10}(0)\|^2_{L^2_w} + \|V_{20}(0)\|^2_{L^2_w} + C_4[1+M(t)]\int_{-\tau}^0 e^{2\mu s}\|V_{10}(s)\|^2_{L^2_w}ds,
\end{aligned}\tag{3.26}$$

for some positive constant $C_4 = \max\{C_1, C_3\} > 0$.

One can find a positive constant $\delta_2 > 0$ such that $C_0^+(\mu) - C_3\delta_2 > 0$. Clearly, δ_2 depends only on β, τ, L and the wave c, because μ depends on these parameters. When $C_0^-(\mu) > 0$, $M(T) \le \delta_2$, i.e.

$$C_0^+(\mu) - C_3M(t) \ge C_0^+(\mu) - C_3M(T) \ge C_0^+(\mu) - C_3\delta_2 > 0,$$

we have

$$e^{2\mu t}\left(\|V_1(t)\|^2_{L^2_w} + \|V_2(t)\|^2_{L^2_w}\right) + \int_0^t e^{2\mu s}\|V_{1\xi}(s)\|^2_{L^2_w}ds \le \|V_{10}(0)\|^2_{L^2_w} + \|V_{20}(0)\|^2_{L^2_w} + C_5\int_{-\tau}^0 e^{2\mu s}\|V_{10}(s)\|^2_{L^2_w}ds, \tag{3.27}$$

for some positive constant $C_5 > 0$. Therefore

$$e^{2\mu t}\|V_i(t)\|^2_{L^2_w} + \int_0^t e^{2\mu s}\|V_{1\xi}(s)\|^2_{L^2_w}ds \le \|V_{10}(0)\|^2_{L^2_w} + \|V_{20}(0)\|^2_{L^2_w} + C_5\int_{-\tau}^0 e^{2\mu s}\|V_{10}(s)\|^2_{L^2_w}ds, \tag{3.28}$$

for $i = 1, 2$.

Similarly, by differentiating (3.2)–(3.3) with respect to ξ, and multiplying the resultant equations by $e^{2\mu t}w(\xi)V_{1\xi}(t,\xi)$, and $e^{2\mu t}w(\xi)V_{1\xi}(t,\xi)$, respectively, and then integrating them over $R\times[0,t]$ with respect to ξ and t, for $t \le T$, we obtain

$$\begin{aligned}
&e^{2\mu t}\left(\|V_{1\xi}(t)\|^2_{L^2_w} + \|V_{2\xi}(t)\|^2_{L^2_w}\right) + \int_0^t e^{2\mu s}\|V_{1\xi\xi}(s)\|^2_{L^2_w}ds \\
&\quad+\int_0^t\int_R e^{2\mu s}\bar{B}^+_{\mu,w}(\xi)w(\xi)V_1^2(s,\xi)d\xi ds + \int_0^t\int_R e^{2\mu s}\bar{B}^-_{\mu,w}(\xi)w(\xi)V_2^2(s,\xi)d\xi ds
\end{aligned}$$

$$
\begin{aligned}
&\le \|V_{1\xi 0}(0)\|_{L_w^2}^2 + \|V_{2\xi 0}(0)\|_{L_w^2}^2 + \int_{-\tau}^{0} e^{2\mu(s+\tau)} |g'(\phi_1(\xi))| \, \|V_{1\xi 0}(s)\|_{L_w^2}^2 ds \\
&\quad + \int_{-\tau}^{0} e^{2\mu(s+\tau)} |g''(\phi_1(\xi))\phi_1'(\xi)| \, \|V_{10}(s)\|_{L_w^2}^2 ds + \int_0^t e^{2\mu(s+\tau)} |g''(\phi_1(\xi))\phi_1'(\xi)| \, \|V_1(s)\|_{L_w^2}^2 ds \\
&\quad + 2\int_0^t \int_R e^{2\mu s} w(\xi) V_{2\xi}(s,\xi) Q_\xi(s-\tau, \xi - c\tau) d\xi ds,
\end{aligned} \tag{3.29}
$$

where the definitions of $\bar{B}^+_{\mu,w}(\xi)$ and $\bar{B}^-_{\mu,w}(\xi)$ can be seen in (3.11) and (3.12). $Q_\xi(s-\tau, \xi - c\tau)$ denotes the differentiation of $Q(s-\tau, \xi - c\tau)$ with respect to ξ.

Using the same analysis as the above for the sixth item of the right-hand-side of (3.29) and combining the basic energy estimate (3.27), we have

$$
\begin{aligned}
&e^{2\mu t}\left(\|V_{1\xi}(t)\|_{L_w^2}^2 + \|V_{2\xi}(t)\|_{L_w^2}^2\right) + \int_0^t e^{2\mu s}\|V_{1\xi\xi}(s)\|_{L_w^2}^2 ds \\
&\le C_6\left(\|V_{10}(0)\|_{H_w^1}^2 + \|V_{20}(0)\|_{H_w^1}^2 + \int_{-\tau}^0 \|V_{10}(s)\|_{H_w^1}^2 ds\right),
\end{aligned} \tag{3.30}
$$

for some positive constant $C_6 > 0$, provided that $M(T) \le \delta_2$. Here we omit the details of the proof.

Combining (3.27) and (3.30), we obtain

$$
\begin{aligned}
&e^{2\mu t}\left(\|V_1(t)\|_{H_w^1}^2 + \|V_2(t)\|_{H_w^1}^2\right) + \int_0^t e^{2\mu s}\|V_{1\xi}(s)\|_{H_w^1}^2 ds \\
&\le C_7\left(\|V_{10}(0)\|_{H_w^1}^2 + \|V_{20}(0)\|_{H_w^1}^2 + \int_{-\tau}^0 \|V_{10}(s)\|_{H_w^1}^2 ds\right),
\end{aligned} \tag{3.31}
$$

for some positive constant $C_7 > 0$, that is independent of T and $V_i(t,\xi)$, $i = 1, 2$, that is to say,

$$
e^{2\mu t}\left(\sum_{i=1}^{2}\|V_i(t)\|_{H_w^1}^2\right) + \int_0^t e^{2\mu s}\|V_{1\xi}(s)\|_{H_w^1}^2 ds \le C_7\left(\|V_{10}(0)\|_{H_w^1}^2 + \|V_{20}(0)\|_{H_w^1}^2 + \int_{-\tau}^0 \|V_{10}(s)\|_{H_w^1}^2 ds\right),
$$

i.e. (3.7) holds for $i = 1, 2$. Then, from (3.7), we automatically reach, for $0 \le t \le T$,

$$
\sum_{i=1}^{2}\|V_i(t)\|_{H_w^1}^2 \le C_7\left(\|V_{10}(0)\|_{H_w^1}^2 + \|V_{20}(0)\|_{H_w^1}^2 + \int_{-\tau}^0 \|V_{10}(s)\|_{H_w^1}^2 ds\right) e^{-2\mu t},
$$

i.e. (3.8) holds. The proof is complete. □

Finally, we prove Theorem 3.1. By Propositions 3.2 and 3.3 and the continuation argument, the proof of Theorem 3.1 is similar to that of Mei and So [48,49]. We only show the outline here.

Proof. Let δ_2, μ, C_7 be positive constants in Proposition 3.3, independent of T. Set

$$
\delta_1 = \max\left\{\sqrt{C_7(1+\tau)}M(0), \delta_2\right\}, \qquad \delta_0 = \min\left\{\frac{\delta_2}{\sqrt{2(1+\tau)}}, \frac{\delta_2}{\sqrt{2C_7}(1+\tau)}\right\}, \tag{3.32}
$$

and

$$
M(0) \le \delta_0 < \delta_2(\le \delta_1). \tag{3.33}
$$

By Proposition 3.2, there exists $t_0 = t_0(\delta_1)$ such that $\mathbf{V}(\xi, t) \in X(-\tau, t_0)$ and

$$
M(t_0) \le \sqrt{2(1+\tau)}M(0) \le \sqrt{2(1+\tau)}\delta_0 \le \delta_2.
$$

On the interval $[0, t_0]$, applying Proposition 3.3, we obtain (3.8) for $t \in [0, t_0]$, and

$$
\begin{aligned}
\sup_{t\in[0,t_0]}\left(\sum_{i=1}^{2}\|V_i(t)\|_{H_w^1}^2\right)^{\frac{1}{2}} &\le \sup_{t\in[0,t_0]}\left\{C_7\left(\|V_{10}(0)\|_{H_w^1}^2 + \|V_{20}(0)\|_{H_w^1}^2 + \int_{-\tau}^0 \|V_{10}(s)\|_{H_w^1}^2 ds\right)\right\}^{\frac{1}{2}} e^{-\mu t} \\
&\le \sqrt{C_7(1+\tau)}M(0) \le \sqrt{C_7(1+\tau)}\delta_0 \le \frac{\delta_2}{\sqrt{2(1+\tau)}}.
\end{aligned} \tag{3.34}
$$

Now, consider the Cauchy problem (3.6) at the initial time $r = t_0$. Using (3.32)–(3.34), we obtain

$$\begin{aligned}
M_{t_0}(0) &= \max\left\{\sup_{t\in[t_0-\tau,t_0]}\|V_1(t)\|_{H_w^1}, \sup\left(\|V_1(t_0)\|_{H_w^1}^2+\|V_2(t_0)\|_{H_w^1}^2\right)^{\frac{1}{2}}\right\}\\
&\le \max\left\{\sup_{t\in[-\tau,0]}\|V_1(t)\|_{H_w^1}, \sup_{t\in[0,t_0]}\|V_1(t)\|_{H_w^1}, \sup_{t\in[0,t_0]}\left(\|V_1(t)\|_{H_w^1}^2+\|V_2(t)\|_{H_w^1}^2\right)^{\frac{1}{2}}\right\}\\
&= \max\left\{\sup_{t\in[-\tau,0]}\|V_1(t)\|_{H_w^1}, \sup_{t\in[0,t_0]}\left(\|V_1(t)\|_{H_w^1}^2+\|V_2(t)\|_{H_w^1}^2\right)^{\frac{1}{2}}\right\}\\
&\le \max\left\{\delta_0, \frac{\delta_2}{\sqrt{2(1+\tau)}}\right\} = \frac{\delta_2}{\sqrt{2(1+\tau)}} \le \delta_1.
\end{aligned}\tag{3.35}$$

Applying Proposition 3.2 once more, we have $\mathbf{V}(t,\xi)\in X(-\tau,2t_0)$ and $M_{t_0}(t_0)\le\sqrt{2(1+\tau)}M_{t_0}(0)$. On the other hand,

$$\begin{aligned}
M_{t_0}(0) &= \max\left\{\sup_{t\in[t_0-\tau,t_0]}\|V_1(t)\|_{H_w^1}, \sup\left(\|V_1(t_0)\|_{H_w^1}^2+\|V_2(t_0)\|_{H_w^1}^2\right)^{\frac{1}{2}}\right\}\\
&\le \max\left\{\sup_{t\in[-\tau,0]}\|V_1(t)\|_{H_w^1}, \sup_{t\in[0,t_0]}\|V_1(t)\|_{H_w^1}, \sup_{t\in[0,t_0]}\left(\|V_1(t)\|_{H_w^1}^2+\|V_2(t)\|_{H_w^1}^2\right)^{\frac{1}{2}}\right\}\\
&= \max\left\{\sup_{t\in[-\tau,0]}\|V_1(t)\|_{H_w^1}, \sup_{t\in[0,t_0]}\left(\|V_1(t)\|_{H_w^1}^2+\|V_2(t)\|_{H_w^1}^2\right)^{\frac{1}{2}}\right\}\\
&\le \max\left\{\delta_0, \frac{\delta_2}{\sqrt{2(1+\tau)}}\right\} = \frac{\delta_2}{\sqrt{2(1+\tau)}}.
\end{aligned}$$

Therefore, we have $M_{t_0}(t_0)\le\delta_2$. Thus,

$$\begin{aligned}
M(2t_0) &= \max\left\{\sup_{t\in[-\tau,0]}\|V_1(t)\|_{H_w^1}, \sup_{[0,2t_0]}\left(\|V_1(t)\|_{H_w^1}^2+\|V_2(t)\|_{H_w^1}^2\right)^{\frac{1}{2}}\right\}\\
&\le \max\left\{\sup_{t\in[-\tau,0]}\|V_1(t)\|_{H_w^1}, \sup_{[0,t_0]}\left(\|V_1(t)\|_{H_w^1}^2+\|V_2(t)\|_{H_w^1}^2\right)^{\frac{1}{2}}, \sup_{[t_0,2t_0]}\left(\|V_1(t)\|_{H_w^1}^2+\|V_2(t)\|_{H_w^1}^2\right)^{\frac{1}{2}}\right\}\\
&\le \max\left\{M(0), \frac{\delta_2}{\sqrt{2(1+\tau)}}, M_{t_0}(t_0)\right\}\\
&\le \max\left\{\delta_0, \frac{\delta_2}{\sqrt{2(1+\tau)}}, \delta_2\right\} = \delta_2.
\end{aligned}$$

We can apply Proposition 3.3 again to obtain (3.8) for $0\le t\le 2t_0$ and

$$\begin{aligned}
\sup_{t\in[0,2t_0]}\left(\sum_{i=1}^{2}\|V_i(t)\|_{H_w^1}^2\right)^{\frac{1}{2}} &\le \sup_{t\in[0,2t_0]}\left\{C_7\left(\|V_{10}(0)\|_{H_w^1}^2+\|V_{20}(0)\|_{H_w^1}^2+\int_{-\tau}^{0}\|V_{10}(s)\|_{H_w^1}^2\,ds\right)\right\}^{\frac{1}{2}}e^{-\mu t}\\
&\le \sqrt{C_7(1+\tau)}M(0)\le\sqrt{C_7(1+\tau)}\delta_0\le\frac{\delta_2}{\sqrt{2(1+\tau)}}.
\end{aligned}$$

Repeating the preceding procedure, we can prove $\mathbf{V}(t,\xi)\in X(-\tau,\infty)$ and the relation (3.8) for all $0\le t<\infty$. Therefore, for $0\le t<\infty$,

$$\|V_i(t)\|_{H_w^1}^2\le C_7\left(\|V_{10}(0)\|_{H_w^1}^2+\|V_{20}(0)\|_{H_w^1}^2+\int_{-\tau}^{0}\|V_{10}(s)\|_{H_w^1}^2\,ds\right)e^{-2\mu t}.\tag{3.36}$$

Also (3.5) follows from (3.36). This completes the proof of Theorem 3.1. □

4. Application

In this section, we shall apply the stability results obtained in Sections 2 and 3 to a special case, i.e. $g(u)=pue^{-au}(e<\frac{p}{\beta}\le e^2)$ in (1.1).

We consider the following epidemic system with delay

$$\begin{cases} \dfrac{\partial}{\partial t}u_1(t,x) = \dfrac{\partial^2}{\partial x^2}u_1(t,x) - u_1(t,x) + u_2(t,x), \\ \dfrac{\partial}{\partial t}u_2(t,x) = -\beta u_2(t,x) + pu_1(t-\tau,x)e^{-au_1(t-\tau,x)}, \end{cases} \tag{4.1}$$

where $e < \frac{p}{\beta} \le e^2$ and $a > 0$, $p > 0$ are constants.

We have the following stability result.

Theorem 4.1. *Assume $\beta e < p \le \min\{\beta e^2, \beta e^{1+\frac{1}{\beta}}, \beta e^{2-\frac{1}{2\beta}}\}$, $\beta > \frac{1}{2}$. For a given traveling wave solution $\mathbf{\Phi}(x+ct) = (\phi_1(x+ct), \phi_2(x+ct))$ of (4.1) with the speed c satisfying*

$$c > \max\left\{c_*, \frac{2\left(p + \frac{2p^2}{a\beta e} + 1 - 2\beta\right)}{c_*}\right\}.$$

If the initial perturbation is

$$\begin{cases} u_{10}(s,x) - \phi_1(x+cs) \in C\left([-\tau,0]; H_w^1(\mathbb{R})\right), \\ u_{20}(x) - \phi_2(x) \in H_w^1(\mathbb{R}) \subset C(\mathbb{R}), \end{cases}$$

where $w(x)$ is the weighted function given in (2.15), then there exist positive constants $\delta_0 = \delta_0(\beta,\tau,g,c)$ and $\mu = \mu(\beta,\tau,g,c)$ such that, when

$$\sup_{s\in[-\tau,0]}\left(\|u_{10}(s,\cdot) - \phi_1(\cdot+cs)\|_{H_w^1(\mathbb{R})} + \|u_{20}(\cdot) - \phi_2(\cdot)\|_{H_w^1(\mathbb{R})}\right) \le \delta_0,$$

the unique solution $(u_1(t,x), u_2(t,x))$ of the Cauchy problem (4.1) and (2.1) exists globally, and satisfies

$$u_i(t,x) - \phi_i(x+ct) \in C\left([0,\infty]; H_w^1(\mathbb{R})\right) \cap L^2\left([0,+\infty); H_w^2(\mathbb{R})\right), \quad i = 1,2$$

and

$$\sup_{x\in\mathbb{R}} |u_i(t,x) - \phi_i(x+ct)| \le Ce^{-\mu t}, \quad t \ge 0,\ i = 1,2.$$

Obviously, (4.1) has two equilibria $\mathbf{u}_- = (0,0)$, and $\mathbf{u}_+ = (K,K) = (\frac{1}{a}\ln\frac{p}{\beta}, \frac{1}{a}\ln\frac{p}{\beta})$. Moreover, it is easy to get that $K^+ = \frac{p}{a\beta e}$ and $K^- = \frac{g'(0)}{\beta}g(K^+) = \frac{p^3}{a\beta^2 e}e^{-\frac{p}{\beta e}}$. The conditions $(C_1) - (C_3)$ always hold under the holding of assumptions $\beta u < pue^{-au} < \frac{2\beta}{a}\ln\frac{p}{\beta}$, $u \in [K^-, K]$ and $\beta u > pue^{-au} > \frac{2\beta}{a}\ln\frac{p}{\beta}$, $u \in [K, K^+]$. Therefore the existence of the traveling waves of (4.1) is guaranteed by Proposition 2.1.

In addition, it is not difficult to examine that $L = \max_{u\in[0,K^+]}|g'(u)| = g'(0) = p$,

$$L_1 = \frac{p}{a\beta e}\max_{u\in[0,K^+]}|g''(u)| = \frac{p}{a\beta e}\max_{u\in[0,K^+]}|p(2-au)e^{-au}| = \frac{2p^2}{a\beta e},$$

and

$$|g'(K)| = \beta\ln\frac{p}{\beta} - \beta < \min\left\{1, \frac{2\beta-1}{2}\right\}.$$

Thus, the proof is similar to Theorem 2.9 (i.e. Theorem 3.1).

Remark 4.2. The global existence and uniqueness of solutions of Cauchy problem (4.1) and (2.1) are guaranteed by Proposition 2.4.

Acknowledgments

Yun-Rui Yang was supported by the NSF of China (11126299). Wan-Tong Li was supported by the NSF of China (11031003). Shi-Liang Wu was supported by the NSF of China (11026127) and the Fundamental Research Funds for the Central Universities (JY10000970005).

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