Pulsating traveling fronts and entire solutions in a discrete periodic system with a quiescent stage

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A B S T R A C T

In this paper, we study pulsating traveling fronts and entire solutions for a spatially discrete periodic reaction–diffusion system with a quiescent stage. Here, the entire solutions are defined in the whole space and for all time $t \in \mathbb{R}$. Under the monostable assumption, the existence and asymptotic behavior of pulsating traveling fronts and spatially periodic solutions connecting two periodic steady states are first established. Combining the leftward and rightward pulsating traveling fronts with different speeds and a spatially periodic solution, the existence and qualitative properties of entire solutions are then proved. Finally, some numerical simulations are carried out to confirm the theoretical results.

1. Introduction

A widely accepted model to describe the evolution of a species population with mobile and non-mobile stages is the following:

$$\begin{cases} 
\partial_t u(x,t) = d_1 u(x,t) + f(u(x,t)) - u(x,t) + \beta v(x,t), \\
\partial_t v(x,t) = \alpha u(x,t) - \beta v(x,t), 
\end{cases}$$

(1.1)

where $(x,t) \in \mathbb{R}^2$, $u$ and $v$ are the densities of the mobile and non-mobile subpopulations, respectively, $d > 0$ is the diffusion coefficient of the mobile subpopulation, $\gamma_1 > 0$ and $\gamma_2 > 0$ are the switching rates and $f$ is the reproduction function. This model describes a species population where the individuals alternate between mobile and stationary states, and only the mobile ones reproduce. Such behavior is typical for invertebrates living in small ponds in arid climates which dry up and reappear subject to rainfall [14]. For more details on the model, we refer to [41–43] and the references cited therein.

It is well known that in reality the natural environments are generally heterogeneous. For example, they are usually mosaic of heterogeneous habitats such as forests, plains, marshes and so on [18]. Therefore, it is very important to understand how heterogeneities influence the ecological dynamics [1,2,7,8,17,31,37]. Clearly, a simple case of heterogeneous environment is the periodic habitat. As an application of the theory in [22], Liang and Zhao considered the spatially periodic version of (1.1)
reaction–advection–diffusion equations in cylinders, and for some reaction–diffusion model systems. More technique of standard monotone iteration scheme to prove the existence and asymptotic behavior of the pulsating traveling fronts using the technique of monotone iteration scheme (Theorem 3.3). Finally, inspired by the works of Hamel and Nadirashvili (Theorem 2.2). Although the existence of such solutions can be obtained by using the monotone dynamical systems theory. Furthermore, the author studied the entire solutions for a class of periodic lattice differential equations with monostable and bistable nonlinearities. For recent works on discrete periodic equations, we refer to [5,10,13,30,39] and the references cited therein.

Throughout this paper, we always assume that the condition (A) holds and use the usual notations for the standard ordering relations 4.7 and 4.8. Finally, in Section 4, some numerical simulations are carried out to confirm the theoretical results.

2. Existence of pulsating traveling fronts

In this section, we study the existence and asymptotic behavior of pulsating traveling fronts connecting 0 and K.

We first consider the following eigenvalue problem associated with a linearization of (1.3) about the trivial steady state 0:

\[
\begin{align*}
\frac{\partial u(x,t)}{\partial t} - \Delta u(x,t) &= f(x,u(x,t), \nabla u(x,t)), \\
\frac{\partial v(x,t)}{\partial t} &= \Delta v(x,t) - b(x)v(x,t),
\end{align*}
\]

where \((x,t) \in \mathbb{R}^2\), \(d(x), \alpha(x)\) and \(\beta(x)\) are continuous, positive, and \(L\)-periodic functions for some \(L > 0\) and \(f(x,\cdot) = f(x + L, \cdot)\). Under the monostable assumption, the existence of spreading speeds and the existence and non-existence of pulsating (or periodic) traveling fronts of (1.2) are established in [22].

When the habitat is a patch environment consisting of all integer nodes of a 1-D lattice, the spatially continuous model (1.2) is reduced to the lattice differential system

\[
\begin{align*}
\frac{d}{dt} u_j(t) &= d_{j,1} u_{j+1}(t) + d_{j,0} u_{j-1}(t) - (d_{j,1} + d_{j,0}) u_j(t) + f_j(u_j(t)) - \alpha_j u_j(t) + \beta_j v_j(t), \\
\frac{d}{dt} v_j(t) &= \alpha_j u_j(t) - \beta_j v_j(t),
\end{align*}
\]

where \((j,t) \in \mathbb{Z} \times [0, \infty)\), \(d_{j,0} > 0\), \(d_{j,1} = d_{j,N} = \alpha_j = \beta_j = \beta_{j,0}\) and \(f_j(\cdot) = f_{j,N}(\cdot)\) for all \(j \in \mathbb{Z}, N\) is a positive integer. In fact, system (1.3) can be viewed as a spatially discrete version of the continuous model (1.2). It is believed that the lattice differential system is more realistic than the corresponding continuous model, since one can only measure the population densities at discrete points. On the other hand, computations for continuous models are often obtained as an approximation of the related discrete ones. For recent works on discrete periodic equations, we refer to [5,10,13,30,39] and the references cited therein.

In this paper, we consider the spatial dynamics of (1.3). The following are monostability assumptions for (1.3):

\[(A) \quad f_j(\cdot) \in C^2([0,1], \mathbb{R}), \quad f_j(0) = f_j(1) = 0, \quad 0 < f_j'(u) \leq f_{j,0}(u) \leq f_j(0)u \quad \text{for all} \quad j \in \mathbb{Z} \quad \text{and} \quad u \in (0,1).\]

Note that the assumption (A) implies that \(f_j'(0) > 0\) for all \(j \in \mathbb{Z}\). One can further see that system (1.3) has two periodic steady states \(0 = \{(0,0)\}_{j \in \mathbb{Z}}\) and \(K = \{K_j\}_{j \in \mathbb{Z}}\) where \(K_j = (1/\beta_j)\). A solution \((\{u_j(t)\}, \{v_j(t)\})\) of (1.3) is said to be a rightward pulsating (or periodic) traveling front connecting 0 and K if it has the form \((u_j(t), v_j(t)) = \Phi_j(-j + ct)\) for a sequences of functions \(\Phi_j(\xi), \xi \in \mathbb{R}\), with

\[
\Phi_{j,N} = \Phi_j(-N), \quad \Phi_j(-\infty) = (0,0) \quad \text{and} \quad \Phi_j(+\infty) = K_j \quad \text{for all} \quad j \in \mathbb{Z}.
\]

Similarly, we say a solution \((\{u_j(t)\}, \{v_j(t)\})\) of (1.3) is a leftward pulsating (or periodic) traveling front connecting 0 and K if it has the form \((u_j(t), v_j(t)) = \Phi_j(j + ct)\) for a sequences of functions \(\Phi_j(\xi), \xi \in \mathbb{R}\), which satisfies (1.4).

Although the pulsating traveling front solution is a key object characterizing the dynamics of periodic lattice differential equations, it is not enough to understand the whole dynamics. In fact, traveling front solutions are only special examples of the so-called entire solutions that are defined in the whole space and for all time \(t \in \mathbb{R}\). Recently, some new types of entire solutions other than traveling fronts are established for various evolution equations with spatially homogeneous environment, see e.g., [3,4,6,9,11,15,16,40,27,21,33] for reaction–diffusion equations with and without delay, [34,35] for delayed lattice differential equations with global interaction, [20,32] for nonlocal dispersal equations, [19,23] for reaction–advection–diffusion equations in cylinders, and [12,28,36,38] for some reaction–diffusion model systems. More recently, Liu and Li [24] and Liu et al. [25] studied the entire solutions of a reaction–advection–diffusion equation with bistable nonlinearity in heterogeneous media and monostable nonlinearity in periodic excitable media, respectively. In [39], the author studied the entire solutions for a class of periodic lattice differential equations with monostable and bistable nonlinearities.

The purpose of this paper is to study the pulsating traveling front and entire solutions of system (1.3). We first establish the existence and asymptotic behavior of rightward and leftward pulsating traveling fronts connecting 0 and K in Section 2 (Theorem 2.2). Although the existence of such solutions can be obtained by using the monotone dynamical systems theory recently developed in [22]. However, these results do not give the exponential decay rate of the pulsating traveling fronts at minus infinity which determine some important properties of the wave. To overcome the shortcoming, we shall use the technique of standard monotone iteration scheme to prove the existence and asymptotic behavior of the pulsating traveling fronts.

In Section 3, the existence and asymptotic behavior of spatially periodic solutions connecting 0 and K are also obtained using the technique of monotone iteration scheme (Theorem 3.3). Finally, inspired by the works of Hamel and Nadirashvili [15] and Wang et al. [34], we construct some new entire solutions other than the pulsating traveling fronts and a spatially periodic solution in Section 4 (Theorem 4.6). Various qualitative features of the entire solutions are also investigated Theorems 4.7 and 4.8. Finally, in Section 4, some numerical simulations are carried out to confirm the theoretical results. Throughout this paper, we always assume that the condition (A) holds and use the usual notations for the standard ordering in \(\mathbb{R}^2\).
\[
\begin{aligned}
    \mu u_{1j} &= d_{j,1} e^{-t} u_{1j+1} + d_{j} e^t u_{1j-1} - (d_{j,1} + d_{j}) u_{1j}, \\
    \mu u_{2j} &= \alpha_j u_{2j+1} - \beta_j u_{2j}, \\
    u_{1j+1} &= u_{1j}, u_{2j+1} = u_{2j}, \\
    & j \in Z.
\end{aligned}
\]  

(2.1)

For convenience, denote

\[
K_{\text{per}} = \left\{ u = (u_{1j}, u_{2j})_{j \in Z} | u_{1j} > 0 \text{ and } u_{1j+N} = u_{j} \text{ for all } j \in Z, \ i = 1, 2 \right\}.
\]

We have the following result.

**Lemma 2.1**

(i) For any $\lambda \in \mathbb{R}$, (2.1) has a principle eigenvalue $M(\lambda)$ which is associated to a strongly positive eigenvector $(\varphi_{1j}(\lambda), \varphi_{2j}(\lambda))_{j \in Z} \in K_{\text{per}}$.

(ii) $M(\cdot)$ is convex in $\mathbb{R}$.

(iii) $M(\lambda) > 0$ for all $\lambda \in \mathbb{R}$, $\lim_{\lambda \to +\infty} \frac{M(\lambda)}{\lambda} = +\infty$ and $\lim_{\lambda \to -\infty} \frac{M(\lambda)}{\lambda} = +\infty$.

(iv) $c_{1} := \frac{M_{c}}{M_{c}} = \inf_{\lambda > 0} \frac{M(\lambda)}{\lambda}$ and $c_{2} := \frac{M_{2}}{M_{2}} = \inf_{\lambda > 0} \frac{M(\lambda)}{\lambda}$ exist and are positive.

(v) For any $c_{1} > c_{1}$, there exists a unique $\lambda_{1} := \lambda_{1}(c_{1}) \in (0, \lambda_{1}^{*})$ such that $M(\lambda_{1}) = c_{1} \lambda_{1}$ and $M(\lambda) < c_{1} \lambda$ for any $\lambda \in (\lambda_{1}, \lambda_{1}^{*}]$.

(vi) For any $c_{2} > c_{2}^{*}$, there exists a unique $\lambda_{2} := \lambda_{2}(c_{2}) \in (0, \lambda_{2}^{*})$ such that $M(\lambda_{2}) = c_{2} \lambda_{2}$ and $M(\lambda) < c_{2} \lambda$ for any $\lambda \in (\lambda_{2}, \lambda_{2}^{*}]$.

**Proof.** Using a generalized Krein–Rutman theorem (see [29]), it is easy to prove that for any $\lambda \in \mathbb{R}$, (2.1) has a principle eigenvalue $M(\lambda)$ which is associated to a strongly positive eigenvector $(\varphi_{1j}(\lambda), \varphi_{2j}(\lambda))_{j \in Z} \in K_{\text{per}}$, i.e.

\[
\begin{aligned}
    M(\lambda) \varphi_{1j}(\lambda) &= d_{j,1} e^{-t} \varphi_{1j+1}(\lambda) + d_{j} e^t \varphi_{1j-1}(\lambda) - (d_{j,1} + d_{j}) \varphi_{1j}(\lambda) + \int_{0}^{\lambda} \varphi_{1j}(\lambda) - \alpha_{j} \varphi_{1j}(\lambda) - \beta_{j} \varphi_{2j}(\lambda), \\
    M(\lambda) \varphi_{2j}(\lambda) &= \alpha_{j} \varphi_{1j}(\lambda) - \beta_{j} \varphi_{2j}(\lambda), \\
    \varphi_{1j,N}(\lambda) &= \varphi_{1j}(\lambda), \varphi_{2j,N}(\lambda) = \varphi_{2j}(\lambda), \\
    & j \in Z.
\end{aligned}
\]  

(2.2)

Similar to [10], Lemma 2.1 (and [13], Lemma 2.2), one can show that $M(\cdot)$ is convex in $\mathbb{R}$.

We now show that $M(\lambda) > 0$ for all $\lambda \in \mathbb{R}$. For any $\lambda \in \mathbb{R}$, let $M(\lambda)$ be the principle eigenvalue of the following eigenvalue problem:

\[
\begin{aligned}
    \mu \varphi_{j} &= d_{j,1} e^{-t} \varphi_{j+1} + d_{j} e^{t} \varphi_{j-1} - (d_{j,1} + d_{j}) \varphi_{j} + \int_{0}^{\lambda} \varphi_{j}, \\
    \varphi_{j,N} &= \varphi_{j}, \\
    & j \in Z.
\end{aligned}
\]  

(2.3)

It follows from [10], Lemma 2.1 that $M(\lambda) > 0$ for $\lambda > 0$. One can easily see that $M(\lambda) = m(-\lambda)$, and hence $m(\lambda) > 0$ for all $\lambda \in \mathbb{R}$. Suppose on the contrary that there exists $\lambda_{1} \in \mathbb{R}$ such that $M(\lambda_{1}) \leq 0$. By (2.2), we have

\[
\begin{aligned}
    M(\lambda_{1}) \varphi_{1j}(\lambda_{1}) &= d_{j,1} e^{-t} \varphi_{1j+1}(\lambda_{1}) + d_{j} e^{t} \varphi_{1j-1}(\lambda_{1}) - (d_{j,1} + d_{j}) \varphi_{1j}(\lambda_{1}) + \int_{0}^{\lambda_{1}} \varphi_{1j}(\lambda_{1}) - \alpha_{j} \varphi_{1j}(\lambda_{1}) - \beta_{j} \varphi_{2j}(\lambda_{1}), \\
    M(\lambda_{1}) \varphi_{2j}(\lambda_{1}) &= \alpha_{j} \varphi_{1j}(\lambda_{1}) - \beta_{j} \varphi_{2j}(\lambda_{1}), \\
    \varphi_{1j,N}(\lambda_{1}) &= \varphi_{1j}(\lambda_{1}), \varphi_{2j,N}(\lambda_{1}) = \varphi_{2j}(\lambda_{1}), \\
    & j \in Z.
\end{aligned}
\]

By a comparison argument, we obtain $M(\lambda_{1}) \geq M(\lambda_{1}) > 0$, a contradiction. Therefore, $M(\lambda) > 0$ for $\lambda \in \mathbb{R}$.

From (2.2), we have

\[
M(\lambda) \varphi_{1j}(\lambda) + \varphi_{2j}(\lambda) = d_{j,1} e^{-t} \varphi_{1j+1}(\lambda) + d_{j} e^{t} \varphi_{1j-1}(\lambda) - (d_{j,1} + d_{j}) \varphi_{1j}(\lambda) + \int_{0}^{\lambda} \varphi_{1j}(\lambda).
\]

(2.4)

Set $\varphi_{1j}(\lambda) = \min_{j \in \{1, \ldots, N\}} \varphi_{1j}(\lambda)$. In view of $M(\lambda) > 0$ for all $\lambda \in \mathbb{R}$, it follows from (2.2) that

\[
\frac{\varphi_{2j}(\lambda)}{\varphi_{1j}(\lambda)} \leq \frac{\lambda_{2}}{\lambda_{1}} \leq \frac{\lambda_{2}}{\lambda_{1}}
\]

for all $j \in \{1, \ldots, N\}$ and $\lambda \in \mathbb{R}$. By (2.4), we have for $\lambda \in \mathbb{R}$,

\[
M(\lambda) \left[ 1 + \frac{\lambda_{2}}{\lambda_{1}} \right] \geq d_{j,1} e^{-t} \varphi_{1j+1}(\lambda) + d_{j} e^{t} \varphi_{1j-1}(\lambda) - (d_{j,1} + d_{j}) \varphi_{1j}(\lambda) + \int_{0}^{\lambda} \varphi_{1j}(\lambda)
\]

\[
\geq d_{j,1} e^{-t} + d_{j} e^{t} - (d_{j,1} + d_{j}) + \int_{0}^{\lambda}
\]

which implies that $\lim_{\lambda \to +\infty} \frac{M(\lambda)}{\lambda} = +\infty$. Similarly, set $\varphi_{1j}(\lambda) = \min_{j \in \{1, \ldots, N\}} \varphi_{1j}(\lambda)$, we can show that

\[
M(-\lambda) \left[ 1 + \frac{\lambda_{2}}{\lambda_{1}} \right] \geq d_{j,1} e^{-t} + d_{j} e^{t} - (d_{j,1} + d_{j}) + \int_{0}^{\lambda}
\]

whence $\lim_{\lambda \to +\infty} \frac{M(-\lambda)}{\lambda} = +\infty$. Therefore, $c_{1} := \frac{M_{c}}{\lambda_{1}} = \inf_{\lambda > 0} \frac{M(\lambda)}{\lambda}$ and $c_{2} := \frac{M_{2}}{\lambda_{2}} = \inf_{\lambda > 0} \frac{M(\lambda)}{\lambda}$ exist and are positive.
Finally, we prove the last two assertions. Let \( g(\lambda) = M(\lambda)/\lambda \) for \( \lambda > 0 \). Using the convexity of the function \( M \), we can show that if \( g(\lambda_1) = g(\lambda_2) = \gamma > c \), for some \( 0 < \lambda_1 < \lambda_2 < \lambda_\ast \), then \( M(\lambda) \geq \gamma \lambda \) for \( \lambda \geq \lambda_2 \), whence \( g(\lambda) \geq \gamma > c \), which is a contradiction. From this, one can show that the function \( g \) is decreasing in \((0, \lambda_\ast^-)\) and the statement (v) holds. Similarly, we can show that (vi) holds. This completes the proof. □

The existence of pulsating traveling fronts is stated as follows.

**Theorem 2.2.** Under the notations above, there holds:

(i) For every \( c_1 \geq c_\ast \), (1.3) has a rightward pulsating traveling front \( \Phi_{c_1}(t) := \{ \Phi_{c_1}(-j + c_1 t) \}_{j \in \mathbb{Z}} = \{(\phi_{c_1,j}(j, \xi), \psi_{c_1,j}(j, \xi))\}_{j \in \mathbb{Z}} \) connecting \( 0 \) and \( K \) which satisfies \( 0 < \phi_{c_1,j}(\xi) < 1 \), \( 0 < \psi_{c_1,j}(\xi) < \alpha_j/\beta_j \), \( \phi_{c_1,j}(\xi) > 0 \) and \( \psi_{c_1,j}(\xi) > 0 \) for all \( j \in \mathbb{Z} \) and \( \xi \in \mathbb{R} \). Moreover, for any \( c_1 > c_\ast \),

\[
\phi_{c_1,j}(\xi), \psi_{c_1,j}(\xi) \leq e^{t(c_1) \xi} \left( \phi_{c_1,j}(\lambda_1(c_1)), \psi_{c_1,j}(\lambda_1(c_1)) \right)
\]

and

\[
\lim_{\xi \to -\infty} \left( \phi_{c_1,j}(\xi), \psi_{c_1,j}(\xi) \right) e^{-t(c_1) \xi} = \left( \phi_{c_1,j}(\lambda_1(c_1)), \psi_{c_1,j}(\lambda_1(c_1)) \right)
\]

for all \( j \in \mathbb{Z} \) and \( \xi \in \mathbb{R} \).

(ii) For every \( c_2 \geq c_\ast \), (1.3) has a leftward pulsating traveling front \( \Phi_{c_2}(t) := \{ \Phi_{c_2}(j + c_2 t) \}_{j \in \mathbb{Z}} = \{(\phi_{c_2,j}(j + c_2 t), \psi_{c_2,j}(j + c_2 t))\}_{j \in \mathbb{Z}} \) connecting \( 0 \) and \( K \) which satisfies \( 0 < \phi_{c_2,j}(\xi) < 1 \), \( 0 < \psi_{c_2,j}(\xi) < \alpha_j/\beta_j \), \( \phi_{c_2,j}(\xi) > 0 \) and \( \psi_{c_2,j}(\xi) > 0 \) for all \( j \in \mathbb{Z} \) and \( \xi \in \mathbb{R} \). Moreover, for any \( c_2 > c_\ast \),

\[
\phi_{c_2,j}(\xi), \psi_{c_2,j}(\xi) \leq e^{t(c_2) \xi} \left( \phi_{c_2,j}(-\lambda_2(c_2)), \psi_{c_2,j}(-\lambda_2(c_2)) \right)
\]

and

\[
\lim_{\xi \to -\infty} \left( \phi_{c_2,j}(\xi), \psi_{c_2,j}(\xi) \right) e^{-t(c_2) \xi} = \left( \phi_{c_2,j}(-\lambda_2(c_2)), \psi_{c_2,j}(-\lambda_2(c_2)) \right)
\]

for all \( j \in \mathbb{Z} \) and \( \xi \in \mathbb{R} \).

To prove Theorem 2.2, we need several lemmas. Substituting \( (u_j(t), v_j(t)) = (\phi_j(\xi), \psi_j(\xi)) \), \( \xi = -j + c_1 t \) into (1.3), we obtain

\[
\begin{align*}
&c_1 \phi_{j+1}(\xi) = d_{j+1} \phi_{j+1}(\xi - 1) + d_{j} \phi_{j} \phi_{j+1}(\xi - 1) - (d_{j} + d_{j+1}) \phi_{j}(\xi) \\
&+ f_{j}(\phi_{j+1}(\xi)) - \alpha_{j+1} \phi_{j+1}(\xi) + \beta_{j+1} \psi_{j+1}(\xi), \quad j \in \mathbb{Z}, \quad \xi \in \mathbb{R}, \\
&c_1 \psi_{j+1}(\xi) = \alpha_{j} \phi_{j}(\xi) - \beta_{j} \psi_{j}(\xi), \quad j \in \mathbb{Z}, \quad \xi \in \mathbb{R}.
\end{align*}
\]

Denote \( W_j = [0,1] \times [0, \alpha_j/\beta_j] \) and

\[
S = \left\{ \phi, \psi \right\} = \{(\phi, \psi)_{j \in \mathbb{Z}} | (\phi_j(\xi), \psi_j(\xi)) \in C(\mathbb{R}, W_j), \phi_j(\xi) = \phi_{j,N}(\xi), \psi_j(\xi) = \psi_{j,N}(\xi) \text{ for all } j \in \mathbb{Z}, \xi \in \mathbb{R} \}.
\]

For any \( (\phi, \psi) \in S \), define an operator \( T = \{ T_j \}_{j \in \mathbb{Z}} = \{ (T_1, T_2) \}_{j \in \mathbb{Z}} \) by

\[
T_j \phi, \psi(\xi) = \int_{-\infty}^{\xi} e^{-t^2} H_j(\phi, \psi)(s) \, ds, \quad i = 1, 2, \quad j \in \mathbb{Z},
\]

where \( L > \max_{x \in [0,1]} \left[ \alpha_j + \beta_j + d_{j+1} + d_{j} \right] + \max_{x \in [0,1]} \left| f_j(u) \right| \right]/c_1 \) and

\[
H_1(\phi, \psi)(\xi) = \frac{1}{c_1} \left[ d_j \phi_{j+1}(\xi - 1) + d_{j+1} \phi_{j}(\xi - 1) - (d_{j} + d_{j+1}) \phi_{j}(\xi) + f_j(\phi_{j}(\xi)) - \alpha_{j} \phi_{j}(\xi) + \beta_{j} \psi_{j}(\xi) \right] + L \phi_{j}(\xi),
\]

\[
H_2(\phi, \psi)(\xi) = \frac{1}{c_1} \left[ \alpha_{j} \phi_{j}(\xi) - \beta_{j} \psi_{j}(\xi) \right] + L \psi_{j}(\xi).
\]

According to the periodicity of \( d_j, \alpha_j, \beta_j \) and \( f_j(\cdot) \), the following observation is straightforward.

**Lemma 2.3**

(i) \( T : S \to S \).

(ii) If \( (\phi_1, \psi_1), (\phi_2, \psi_2) \in S \) with \( (\phi_1, \psi_1) \geq (\phi_2, \psi_2) \), then \( T(\phi_1, \psi_1) \geq T(\phi_2, \psi_2) \).

(iii) If \( (\phi, \psi) \in S \) is non-decreasing, then so is \( T(\phi, \psi) \).

(iv) \( (\phi, \psi) \in S \) solves (2.9) if and only if it satisfies \( (\phi, \psi) = T(\phi, \psi) \).
Lemma 2.5. Assume that (2.9) admits a supersolution \( \Phi = \{\Phi_j\}_{j \in \mathbb{Z}} \in S \) and a subsolution \( \Phi = \{\Phi_j\}_{j \in \mathbb{Z}} = \{(\phi_j, \psi_j)\}_{j \in \mathbb{Z}} \) which satisfy

(i) \( \Phi_j(\xi) \) is non-decreasing in \( \mathbb{R} \), \( \Phi_j(-\infty) = (0, 0) \), and \( \Phi_j(\infty) = K_j \) for \( j \in \mathbb{Z} \);

(ii) \( \phi_j \not\equiv 0, \psi_j \not\equiv 0 \) and \( 0 \leq \Phi_j(\xi) \leq \Phi_j(\xi) \leq K_j \) for all \( j \in \mathbb{Z} \) and \( \xi \in \mathbb{R} \).

Then (2.9) has a non-decreasing solution \( \Phi = \{\Phi_j\}_{j \in \mathbb{Z}} \in S \) which satisfies \( \Phi(-\infty) = (0, 0) \) and \( \Phi_j(\infty) = K_j \) for all \( j \in \mathbb{Z} \), that is, (1.3) has a rightward pulsating traveling front connecting 0 and K.

Proof. Consider the monotone iteration scheme starting from the supersolution \( \Phi = \Phi^\infty \in S \):

\[
\begin{align*}
\Phi^{n+1} &= T\Phi^n, & n \in \mathbb{N}, \\
\Phi^{(1)} &= \Phi = \Phi^\infty.
\end{align*}
\] (2.12)

Using Lemma 2.3, it is easy to show that (2.9) has a non-decreasing solution \( \Phi = \{\Phi_j\}_{j \in \mathbb{Z}} = \{(\phi_j, \psi_j)\}_{j \in \mathbb{Z}} \in S \) with \( \Phi_j(\xi) \leq \Phi_j(\xi) \leq \Phi_j(\xi) \) for \( \in \mathbb{Z} \). Since \( \Phi(-\infty) = (0, 0) \), \( \Phi_j(\infty) = (0, 0) \) for all \( j \in \mathbb{Z} \).

Now, we show \( \Phi_j(\infty) = K_j \) for all \( j \in \mathbb{Z} \). In view of \( \phi_j \not\equiv 0, \psi_j \not\equiv 0 \), we have \( \Phi_j(\infty) \) is non-decreasing, we have \( \Phi_j(\infty) \) exists and \( \phi_j(\infty) \in (0, 1) \) and \( \psi_j(\infty) \in (0, K) \). By l'Hospital's rule, for any \( j \in \mathbb{Z} \),

\[
\phi_j(\infty) = \lim_{\xi \to \infty} T_j(\phi_j, \psi_j)(\xi) = \lim_{\xi \to \infty} \frac{1}{L} H_j(\phi_j, \psi_j)(\xi)
\]

which implies that

\[
d_{j+1} \phi_j(\xi) + d_j \phi_j(\xi) - (d_{j+1} + d_j) \psi_j(\xi) + f_j(\psi_j(\xi)) - \alpha \phi_j(\xi) + \beta \psi_j(\xi) = 0.
\] (2.13)

Similarly, there holds

\[
\phi_j(\xi) - \beta \psi_j(\xi) = 0.
\] (2.14)

It then follows from (2.13) and (2.14) that

\[
d_{j+1} \phi_j(\xi) + d_j \phi_j(\xi) - (d_{j+1} + d_j) \psi_j(\xi) + f_j(\psi_j(\xi)) = 0.
\]

Set \( \phi_j(\infty) = \min_{j \in \mathbb{Z}} \phi_j(\infty) \). Then we have

\[
0 = d_{j+1} \phi_j(\xi) + d_j \phi_j(\xi) - (d_{j+1} + d_j) \psi_j(\xi) + f_j(\psi_j(\xi)) \geq f_j(\psi_j(\xi)) \geq 0,
\]

and hence \( f_j(\psi_j(\xi)) = 0 \). Then, \( \phi_j(\xi) = 1 \) for all \( j \in \mathbb{Z} \). From (2.14), \( \psi_j(\xi) = \alpha_j / \beta_j \) for all \( j \in \mathbb{Z} \). Therefore, \( \Phi_j(\infty) = K_j \) for \( j \in \mathbb{Z} \). This completes the proof. □

Let \( c_1 > c_1 \) be any given number. Choose \( v \in (\xi_1, \min(2\xi_1, \xi_2^*)) \) such that \( M(v) < c_1 v \). Take \( L_1 = \max_{j \in \mathbb{Z} \times [0, 1]} |f_j'(u)| \) and

\[
q > \max_{j \in \mathbb{Z}} \max \left\{ \frac{\phi_j'(\xi_1)}{\phi_j(v)}, \frac{L_1 \phi_j'^2(\xi_1)}{2c_1 v - M(v) \phi_j(v)} \right\},
\]

where \( (\phi_1, \phi_2, \psi_1, \psi_2) \in K_{per} \) and \( (\phi_1(\xi_1), \phi_2(\xi_1)) \) are the eigenvectors corresponding to the principle eigenvalue \( M(v) \) and \( M(\xi_1) \) of (2.1), respectively.

Define two functions \( \Phi(\xi) = \{\Phi_j(\xi)\}_{j \in \mathbb{Z}} = \{(\phi_j(\xi), \psi_j(\xi))\}_{j \in \mathbb{Z}} \) and \( \Phi(\xi) = \{\Phi_j(\xi)\}_{j \in \mathbb{Z}} = \{(\phi_j(\xi), \psi_j(\xi))\}_{j \in \mathbb{Z}} \) as follows:

\[
\Phi_j(\xi) = \min \left\{ 1, e^{c_1 \xi} \phi_j(\xi_1) \right\}, \quad \phi_j(\xi) = \max \left\{ 0, e^{c_1 \xi} \phi_j(\xi_1) - q e^{c_1 \xi} \phi_1(v) \right\},
\]

\[
\Phi_j(\xi) = \min \left\{ \frac{\phi_j}{\beta_j}, e^{c_1 \xi} \psi_j(\xi_1) \right\}, \quad \psi_j(\xi) = \max \left\{ 0, e^{c_1 \xi} \phi_j(\xi_1) - q e^{c_1 \xi} \phi_2(v) \right\}.
\]
Lemma 2.6 
(i) $\Phi \in \mathcal{S}$ and $0 \leq \Phi_j(\xi) \leq \Phi(\xi) \leq K_j$ for all $j \in \mathbb{Z}$ and $\xi \in \mathbb{R}$.
(ii) $\Phi$ and $\Phi_j$ are supersolution and subsolution of (2.9), respectively.

Proof. It is easy to see that $\Phi \in \mathcal{S}$ and $0 \leq \Phi_j(\xi) \leq \Phi(\xi) \leq K_j$ for all $j \in \mathbb{Z}$. We now prove that $\Phi$ is a supersolution. For convenience, we denote $\Phi = \{\phi_j, \psi_j\}_{j \in \mathbb{Z}}$ and

$$
F_{1j}(\Phi)(\xi) = c_1 \phi_j^3(\xi) - d_{j+1} \phi_{j}^3(\xi - 1) - d_j \phi_{j+1}^3(\xi + 1) + (d_{j+1} + d_j) \phi_j(\xi) - f_j(\phi_j(\xi)) + \alpha_j \phi_j(\xi) - \beta_j \psi_j(\xi),
$$

$$
F_{2j}(\Phi)(\xi) = c_1 \psi_j^3(\xi) - \alpha_j \phi_j(\xi) + \beta_j \psi_j(\xi).
$$

Take $\hat{\xi}_j = - \frac{1}{c_1} \log \phi_j(\lambda_j)$. For $\xi \geq \hat{\xi}_j$, $\phi_j(\xi) = 1$, and hence,

$$
F_{1j}(\Phi)(\xi) = -d_{j+1} \phi_{j}^3(\xi - 1) - d_j \phi_{j+1}^3(\xi + 1) + (d_{j+1} + d_j) \phi_j(\xi) + \alpha_j \phi_j(\xi) - \beta_j \psi_j(\xi) \geq 0.
$$

For $\xi < \hat{\xi}_j$, $\phi_j(\xi) = e^{c_1(\xi - \lambda_j)} \phi_j(\lambda_j)$. Noting that $M(\lambda_j) = c_1 \lambda_j$, there holds

$$
F_{1j}(\Phi)(\xi) \geq e^{c_1(\xi - \lambda_j)} \left[ c_1 \lambda_j \phi_j(\lambda_j) - d_{j+1} e^{c_1(\xi - \lambda_j)} \phi_j(\lambda_j - 1) - d_j e^{c_1(\xi - \lambda_j)} \phi_j(\lambda_j + 1) + (d_{j+1} + d_j) \phi_j(\lambda_j) - f_j(\phi_j(\lambda_j) + \alpha_j \phi_j(\lambda_j) - \beta_j \psi_j(\lambda_j) \right]
$$

$$
= e^{c_1(\xi - \lambda_j)} \left[ c_1 \lambda_j \phi_j(\lambda_j) - M(\lambda_j) \phi_j(\lambda_j) \right] \geq 0.
$$

Then $F_{1j}(\Phi)(\xi) \geq 0 \text{ a.e. in } \mathbb{R}$ for all $j \in \mathbb{Z}$. Similarly, we can show that $F_{2j}(\Phi)(\xi) \geq 0 \text{ a.e. in } \mathbb{R}$ for all $j \in \mathbb{Z}$. Therefore, $\Phi$ is a supersolution of (2.9).

Next, we prove $\Phi$ is a subsolution of (2.9). Let $\eta_j = \frac{1}{c_1} \log \phi_j(\lambda_j) < 0$. For $\xi > \eta_j$, $\phi_j(\xi) = 0$. Then, we have

$$
F_{1j}(\Phi)(\xi) = -d_{j+1} \phi_{j}^3(\xi - 1) - d_j \phi_{j+1}^3(\xi + 1) + \beta_j \psi_j(\xi) \leq 0.
$$

For $\eta_j < \xi < \lambda_j$, $\phi_j(\xi) = e^{c_1(\xi - \lambda_j)} \phi_j(\lambda_j)$. Note that $f_j(u) \geq f_j(0)u - \frac{1}{2} u^2$ for all $j \in \mathbb{Z}$ and $u \in [0, 1]$. Then for $\xi > \eta_j$, we have

$$
F_{1j}(\Phi)(\xi) \leq e^{c_1(\xi - \lambda_j)} \left[ c_1 \lambda_j \phi_j(\lambda_j) - d_{j+1} e^{c_1(\xi - \lambda_j)} \phi_j(\lambda_j - 1) - d_j e^{c_1(\xi - \lambda_j)} \phi_j(\lambda_j + 1) + (d_{j+1} + d_j) \phi_j(\lambda_j) - f_j(\phi_j(\lambda_j) + \alpha_j \phi_j(\lambda_j) - \beta_j \psi_j(\lambda_j) \right]
$$

$$
- \beta_j \psi_j(\lambda_j) - q e^{c_1(\xi - \lambda_j)} \left[ \phi_j(\lambda_j - 1) - \phi_j(\lambda_j + 1) \right] + \frac{L_1}{2} \frac{d^2 \phi_j}{d \xi^2}(\lambda_j) \leq e^{c_1(\xi - \lambda_j)} \left[ c_1 \lambda_j \phi_j(\lambda_j) - q \phi_j(\lambda_j) \right] \leq 0.
$$

Thus $F_{1j}(\Phi)(\xi) \leq 0 \text{ a.e. in } \mathbb{R}$ for all $j \in \mathbb{Z}$. Similarly, we can show that $F_{2j}(\Phi)(\xi) \leq 0 \text{ a.e. in } \mathbb{R}$ for all $j \in \mathbb{Z}$. Therefore, $\Phi$ is a subsolution of (2.9). The proof is complete. □

Proof of Theorem 2.2. From Lemmas 2.5 and 2.6, we conclude that for every $c_1 > c^*_1$, (1.3) has a non-decreasing rightward pulsating traveling front $\Phi_j(\cdot - c_1 t) := (\phi_j, \psi_j) \in \mathcal{S}$ connecting $0$ and $K$ and satisfying (2.5) and (2.6).

The existence of non-decreasing rightward pulsating traveling wave fronts when $c = c_1$ could be obtained by a limiting argument similar to that of [13]. We omit the details.

Now, we show that $0 < \psi_{j1}(\xi) < 1$ and $0 < \psi_{j2}(\xi) < \alpha_j / \beta_j$ for all $j \in \mathbb{Z}$ and $\xi \in \mathbb{R}$. Clearly, $(\phi_{j1}(\cdot), \psi_{j2}(\cdot)) \in [0, 1] \times [0, \alpha_j / \beta_j]$. We first show $\phi_{j1}(\xi) > 0$. Suppose on the contrary that there exists $j_0, \xi_0 \in \mathbb{Z} \times \mathbb{R}$ that $\phi_{j_0}(\xi_0) = 0$. Then, $\phi_{j_0}(\xi_0) = 0$. From the second equation of (2.9), we have

$$
0 \leq \phi_{j_0}(\xi_0) = 0.
$$

which implies that $\psi_{j_0}(\xi_0) = 0$. By the first equation of (2.9), we obtain

$$
0 = d_{j_0+1} \phi_{j_0+1}(\xi_0 - 1) + d_{j_0} \phi_{j_0+1}(\xi_0 + 1) \geq 0,
$$

and hence $\phi_{j_0+1}(\xi_0 + 1) = 0$. Repeat the produce, we get $\phi_{j_0-n}(\xi_0 + n) = 0$ for all $n \in \mathbb{N}$. In view of $\phi_{j_0+n}(\cdot) = \phi_{j_0}(\cdot)$ for all $j \in \mathbb{Z}$, we have $\phi_{j_0}(\xi_0 + nN) = 0$ for all $n \in \mathbb{N}$ which contradicts to $\phi_{j_0}(\cdot) = 1$. Thus, $\phi_{j1}(\xi) > 1$ for all $j \in \mathbb{Z}$ and $\xi \in \mathbb{R}$. According to

$$
\psi_{j1}(\xi) = \frac{\alpha_j}{c_1} \int_{-\infty}^{\xi} e^{-c_1(\xi - s)} \phi_{j1}(s)ds,
$$

we have $\psi_{j1}(\xi) > 0$ for all $j \in \mathbb{Z}$ and $\xi \in \mathbb{R}$. Similarly, we can show that $\phi_{j1}(\xi) < 1$ and $\psi_{j2}(\xi) < \alpha_j / \beta_j$ for all $j \in \mathbb{Z}$ and $\xi \in \mathbb{R}$.

Next, we show that $\phi_{j1}(\xi) > 0$ and $\psi_{j1}(\xi) > 0$ for all $j \in \mathbb{Z}$ and $\xi \in \mathbb{R}$. Note that for all $j \in \mathbb{Z}$ and $\xi \in \mathbb{R}$,
Similar to Lemma 2.6, we have the following result.

\[
c_1 \phi'_{ij}(\xi) = d_{j,1} \phi'_{ij+1}(\xi - 1) + d_{j} \phi'_{i,j-1}(\xi + 1) - (d_{j,1} + d_{j}) \phi'_{ij}(\xi) + f_j(\phi_{ij}(\xi)) \phi'_{ij}(\xi) - \alpha_j \phi_{ij}(\xi) + \beta_j \psi_{ij}(\xi) \geq \left[ f_j(\phi_{ij}(\xi)) \right] -d_{j,1} - d_{j} \phi'_{ij}(\xi) \geq c_1 m_1 \phi_{ij}(\xi) - c_2 \phi_{ij}(\xi) - \alpha_j \phi_{ij}(\xi) - \beta_j \psi_{ij}(\xi) \geq -c_1 m_2 \psi_{ij}(\xi),
\]

where \( c_1 m_1 = \min \{0,1/\xi \geq 0.1 \} f_j(u) - \max_{\xi \in Z} \{d_{j,1} + d_{j} \} \) and \( c_1 m_2 = \max_{\xi \in Z} \beta_j \). Suppose that there exists \( (j_2, \xi) \in Z \times \mathbb{R} \) that \( \phi_{1,j_2}(\xi_2) = 0 \). From (2.15), we obtain \( 0 = \phi'_{1,j_2}(\xi_2) \geq \phi'_{1,j_2}(\xi) e^{-\min(2-i)} \geq 0 \) for all \( \xi \leq \xi_2 \), which yields that \( \phi_{1,j_2}(\xi) = 0 \) for all \( \xi \leq \xi_2 \). Thus, \( \phi_{1,j_2}(\xi) = \phi_{1,j_2}(\xi_2) \) for all \( \xi \leq \xi_2 \). Letting \( \xi \to -\infty \), we get \( \phi_{1,j_2}(\xi_2) = 0 \) which contradicts to \( \phi_{1,j_2}(\xi_2) > 0 \). Therefore, \( \phi_{1,j_2}(\xi) > 0 \) for all \( j \in Z \) and \( \xi \in \mathbb{R} \). Similarly, one can easily show that \( \psi_{1,j_2}(\xi) > 0 \) for all \( j \in Z \) and \( \xi \in \mathbb{R} \). The existence and asymptotic behavior of the leftward pulsating traveling front can be proved by a little modification. We omit it here.

Now we complete the proof of Theorem 2.2.

3. Existence of spatially periodic solutions

In this section, we study the spatially periodic solutions of system (1.3) connecting \( \Theta \) and \( K \), that is, solutions of the problem:

\[
\begin{align*}
\begin{cases}
    u_j(t) = d_{j,1} u_{j+1}(t) + d_{j} u_{j-1}(t) - (d_{j,1} + d_{j}) u_j(t) + f_j(u_j(t)) - \alpha_j u_j(t) - \beta_j v_j(t), & j \in Z, t \in \mathbb{R}, \\
    v_j(t) = \alpha_j u_j(t) - \beta_j v_j(t), & j \in Z, t \in \mathbb{R},
\end{cases}
\end{align*}
\]

(3.1)

Let \( \lambda^* = M(0) \) and \( \varphi(0) = (\phi_{1,j_0}(0), \phi_{2,j_0}(0)) \in K_{per} \), where \( \varphi(0) \) is the eigenvalue corresponding to the principle eigenvalue \( M(0) \) of (2.1) with \( \lambda = 0 \).

For any \( \phi, \psi \in S \), define an operator \( G = \{ G_j \}_{j \in Z} = \{ (G_{ij}, G_{2j}) \}_{j \in Z} \) by

\[
G_j(\phi, \psi)(t) = \int_{-\infty}^t e^{\mu(t-s)} P_j(\phi, \psi)(s) ds, \quad i = 1, 2, \quad j \in Z,
\]

where \( S \) is defined by (2.10), \( \mu > \max_{\xi \in Z} \{ \alpha_j + \beta_j + d_{j,1} + d_j \} \) and

\[
\begin{align*}
P_{1j}(\phi, \psi)(t) &= d_{j,1} \phi_{j+1}(t - 1) + d_{j} \phi_{j-1}(t + 1) - (d_{j,1} + d_{j}) \phi_j(t) + f_j(\phi_j(t)) - \alpha_j \phi_j(t) - \beta_j \psi_j(t) + \mu \phi_j(t), \\
P_{2j}(\phi, \psi)(t) &= \alpha_j \phi_j(t) - \beta_j \psi_j(t) + \mu \psi_j(t).
\end{align*}
\]

Similar to Lemma 2.3, the following result hold.

**Lemma 3.1**

(i) \( G : S \to S \).

(ii) If \( (\phi_1, \psi_1), (\phi_2, \psi_2) \in S \) with \( (\phi_1, \psi_1) \geq (\phi_2, \psi_2) \), then \( G(\phi_1, \psi_1) \geq G(\phi_2, \psi_2) \).

(iii) If \( (\phi, \psi) \in S \) is non-decreasing, then so is \( G(\phi, \psi) \).

(iv) \( (\phi, \psi) \in S \) solves the problem (3.1) if and only if it satisfies \( (\phi, \psi) = G(\phi, \psi) \).

Define supersolution and subsolution of (3.1) in a similar way as in Definition 2.4. For any fixed \( v_1 \in (1, 2] \) and sufficiently large \( q_1 > 1 \), define two functions \( \Phi^+(t) = \{ \phi^+_j(t) \}_{j \in Z} = \{ (\phi^+_j(t), \psi^+_j(t)) \} \) and \( \Phi^-(t) = \{ \phi^-_j(t) \}_{j \in Z} = \{ (\phi^-_j(t), \psi^-_j(t)) \} \) as follows:

\[
\begin{align*}
\phi^+_j(t) &= \min \left\{ 1, e^{v_1 t} \phi_{1j}(0) \right\}, \quad \phi^-_j(t) = \max \left\{ 0, e^{v_1 t} \phi_{1j}(0) - q_1 e^{v_1 t} \phi_{1j}(0) \right\},
\psi^+_j(t) &= \min \left\{ \frac{\alpha_j}{\beta_j} e^{v_1 t} \phi_{2j}(0) \right\}, \quad \psi^-_j(t) = \max \left\{ 0, e^{v_1 t} \phi_{2j}(0) - q_1 e^{v_1 t} \phi_{2j}(0) \right\}.
\end{align*}
\]

Similar to Lemma 2.6, we have the following result.

**Lemma 3.2**

(i) \( \Phi^-, \Phi^+ \in S \) and \( 0 \leq \Phi^+_j(\xi) \leq \Phi^-_j(\xi) \leq K_j \) for all \( j \in Z \) and \( \xi \in \mathbb{R} \).

(ii) \( \Phi^+ \) and \( \Phi^- \) are supersolution and subsolution of (3.1), respectively.
Theorem 3.3. There exists a solution $\Gamma(t) = \{\Gamma_j(t)\}_{j \in \mathbb{Z}}$ of (3.1) which satisfies $\Gamma_j(t) = K_j$ and $\lim_{t \to +\infty} \Gamma_j(t) e^{-\lambda t} = (\phi_j(0), \phi_j(0))$ for all $j \in \mathbb{Z}$. Moreover, there holds $\Gamma_j(t) \leq e^{\lambda t} (\phi_j(0), \phi_j(0))$, $0 < \Gamma_j(t) < 1$, $0 < \Gamma_{2j}(t) < \frac{1}{\beta_j}$ and $\Gamma_{2j}(t) > 0$ for $j = 1, 2, j \in \mathbb{Z}, t \in \mathbb{R}$.

Proof. Using the technique of monotone iteration scheme, the proof is similar to that of Theorem 2.2. Here, we omit the details. $\square$

4. Existence and qualitative properties of entire solutions

In this section, we first give some preliminaries. The existence of entire solutions is then proved by using comparison principle. Finally, some qualitative properties of the entire solution are investigated.

4.1. Preliminaries

Let $t_0 \in \mathbb{R}$ be any given constant. Consider the initial value problem:

$$
\begin{align*}
\frac{d}{dt} u_j(t) &= d_{j+1} u_{j+1}(t) + d_j u_{j-1}(t) - (d_{j+1} + d_j) u_j(t) + f_j(u_j(t)) - \alpha_j u_j(t) + \beta_j v_j(t), \quad j \in \mathbb{Z}, \quad t > t_0, \\
\frac{d}{dt} v_j(t) &= \alpha_j u_j(t) - \beta_j v_j(t), \quad j \in \mathbb{Z}, \quad t > t_0, \\
(u_j(t_0), v_j(t_0)) &= (u_j^0, v_j^0), \quad j \in \mathbb{Z}.
\end{align*}
$$

(4.1)

Recall that $W_j = [0, 1] \times [0, \alpha_j / \beta_j]$.

Definition 4.1. A sequence of differential functions $\{(u_j(t), v_j(t))\}_{j \in \mathbb{Z}}$ with $(u_j, v_j) \in C([t_0, \infty), W_j)$ is called a supersolution of (4.1) on $[t_0, \infty)$ if

$$
\begin{align*}
\frac{d}{dt} u_j(t) &\geq d_{j+1} u_{j+1}(t) + d_j u_{j-1}(t) - (d_{j+1} + d_j) u_j(t) + f_j(u_j(t)) - \alpha_j u_j(t) + \beta_j v_j(t), \\
\frac{d}{dt} v_j(t) &\geq \alpha_j u_j(t) - \beta_j v_j(t)
\end{align*}
$$

for all $j \in \mathbb{Z}$ and $t > t_0$. A supersolution of (4.1) is defined by reversing the inequality.

By Definition 4.1, We have the following result. Its proof is standard and omitted, see e.g. [26, Lemma 4.1].

Lemma 4.2

(i) For any $\phi^0 = (u_j^0, v_j^0)_{j \in \mathbb{Z}}$ with $(u_j^0, v_j^0) \in C([t_0, \infty), W_j)$, (4.1) admits a unique solution $w(t; \phi^0) = \{(u_j(t), v_j(t))\}_{j \in \mathbb{Z}}$ on $[t_0, +\infty)$ which satisfies $(u_j(t_0), v_j(t_0)) = (u_j^0, v_j^0)$ and $(u_j, v_j) \in C([t_0, \infty), W_j)$ for all $j \in \mathbb{Z}$.

(ii) Let $\{(u_j^+(t), v_j^+(t))\}_{j \in \mathbb{Z}}$ and $\{(u_j^-(t), v_j^-(t))\}_{j \in \mathbb{Z}}$ be supersolution and subsolution of (4.1), respectively. If $(u_j^+(t_0), v_j^+(t_0)) \geq (u_j^-(t_0), v_j^-(t_0))$ for all $j \in \mathbb{Z}$, then $(u_j^+(t), v_j^+(t)) \geq (u_j^-(t), v_j^-(t))$ for all $j \in \mathbb{Z}$ and $t \geq t_0$.

Moreover, we give the prior estimate of solutions of (1.3) in the following lemma. Its proof is easy so we omit it.

Lemma 4.3. Assume that $\{(u_j(t; \phi^0), v_j(t; \phi^0))\}_{j \in \mathbb{Z}}$ is a solution of (1.3) with initial value $\phi^0 = (u_j^0, v_j^0)_{j \in \mathbb{Z}}$ satisfying $(u_j^0, v_j^0) \in C([t_0, \infty), W_j)$, then there exists a positive constant $M$, independent of $t_0$ and $\phi^0$, such that for any $j \in \mathbb{Z}$ and $t > t_0 + 1$,

$$
\left| \frac{d}{dt} u_j(t; \phi^0) \right| \leq M, \quad \left| \frac{d^2}{dt^2} u_j(t; \phi^0) \right| \leq M, \quad \left| \frac{d}{dt} v_j(t; \phi^0) \right| \leq M \quad \text{and} \quad \left| \frac{d^2}{dt^2} v_j(t; \phi^0) \right| \leq M.
$$

We also need following result in the sequel.

Lemma 4.4. Let $w_j^+ = (u_j^+, v_j^+) \in C^1([t_0, +\infty), [0, +\infty) \times [0, +\infty))$ and $w_j^- = (u_j^-, v_j^-) \in C^1([t_0, +\infty), (-\infty, 1) \times (-\infty, \alpha_j / \beta_j))$ be such that $w_j^+(t_0) \geq w_j^-(t_0)$ for all $j \in \mathbb{Z}$, and

$$
\begin{align*}
\frac{d}{dt} u_j^+(t) &\geq d_{j+1} u_{j+1}(t) + d_j u_{j-1}(t) - (d_{j+1} + d_j) u_j(t) + f_j(u_j(t)) - \alpha_j u_j(t) + \beta_j v_j(t), \\
\frac{d}{dt} v_j^+(t) &\geq \alpha_j u_j(t) - \beta_j v_j(t),
\end{align*}
$$

and

$$
\begin{align*}
\frac{d}{dt} u_j^-(t) &\leq d_{j+1} u_{j+1}(t) + d_j u_{j-1}(t) - (d_{j+1} + d_j) u_j(t) + f_j(u_j(t)) - \alpha_j u_j(t) + \beta_j v_j(t), \\
\frac{d}{dt} v_j^-(t) &\leq \alpha_j u_j(t) - \beta_j v_j(t).
\end{align*}
$$

(4.3)
\[
\begin{align*}
\frac{d}{dt}u_j^*(t) &\leq d_{j-1}u_{j-1}^*(t) + d_ju_{j=1}^*(t) - (d_{j-1} + d_j)u_j^*(t) \\
-\frac{1}{f_j^*(0)}u_j^*(t) &\leq \beta_ju_j^*(t), \\
\frac{d}{dt}v_j^*(t) &\leq \gamma_ju_j^*(t) - \beta_jv_j^*(t),
\end{align*}
\] (4.4)

for \(j \in \mathbb{Z}\) and \(t > t_0\). Then \(W_j^*(t) \geq W_j^*(t)\) for all \(j \in \mathbb{Z}\) and \(t \geq t_0\).

4.2. Existence of entire solutions

In this section, we will use the properties of previous sections to obtain an appropriate upper estimate for solutions of (1.3) and then prove the existence of entire solutions.

For any \(n \in \mathbb{Z}, h_1, h_2, h_3 \in \mathbb{R}, c_1 > c^*_2, c_2 > c^*_1\), and \(\lambda_1, \lambda_2, \lambda_3 \in (0, 1)\) with \(\lambda_1 + \lambda_2 + \lambda_3 \geq 2\), denote

\[
\phi_n^0 := \max \{\chi_1\Phi_1(-j - c_1n + h_1), \chi_2\Phi_2(j - c_2n + h_2), \chi_3\Gamma_j(-n + h_3)\},
\]

\[
W_j^*(t) := \max \{\chi_1\Phi_1(-j + c_1(t + h_1), \chi_2\Phi_2(j + c_2t + h_2), \chi_3\Gamma_j(t + h_3)\},
\]

where \(j \in \mathbb{Z}\) and \(t \in \mathbb{R}\). Let \(W_n^*(t) = \left\{W_j^*(t)\right\}_{j \in \mathbb{Z}} := \left\{(U_j^*(t), V_j^*(t))\right\}_{j \in \mathbb{Z}}\) be the unique solution of the following initial value problem

\[
\begin{align*}
\frac{d}{dt}U_j^*(t) &= d_{j-1}U_{j-1}^*(t) + d_jU_{j=1}^*(t) - (d_{j-1} + d_j)U_j^*(t) \\
+ f_j^*(0)U_j^*(t) - \beta_jU_j^*(t), & j \in \mathbb{Z}, t > -n, \\
\frac{d}{dt}V_j^*(t) &= \gamma_jU_j^*(t) - \beta_jV_j^*(t), & j \in \mathbb{Z}, t > -n,
\end{align*}
\] (4.5)

Then, from Lemma 4.2, \(W_j^*(t) \leq W_j^{n-1}(t) \leq K_j\) for all \(j \in \mathbb{Z}\) and \(t \geq -n\). The following result gives an appropriate upper estimate of \(W_n^*(t)\).

**Lemma 4.5.** The function \(W_n^*(t) = \left\{W_j^*(t)\right\}_{j \in \mathbb{Z}}\) satisfies

\[
W_j^*(t) \leq \min \{K_j, \Pi_1(j, t), \Pi_2(j, t), \Pi_3(j, t)\}
\]

for all \(j \in \mathbb{Z}\) and \(t \geq -n\), where

\[
\begin{align*}
\Pi_1(j, t) &= \chi_1\Phi_1(-j + c_1(t + h_1)), \\
\Pi_2(j, t) &= \chi_2\Phi_2(j + c_2t + h_2), \\
\Pi_3(j, t) &= \chi_3\Gamma_j(t + h_3).
\end{align*}
\]

**Proof.** The proof is similar to that of Lemma 5.2 [34] and thus omitted. □

The existence result of entire solutions of (1.3) is stated as follows.

**Theorem 4.6.** Let (A) hold. Assume further that \(f_j^*(u) \leq f_j^*(0)\) for all \((j, u) \in \mathbb{Z} \times [0, 1]\). For any \(h_1, h_2, h_3 \in \mathbb{R}, c_1 > c^*_2, c_2 > c^*_1\), and \(\lambda_1, \lambda_2, \lambda_3 \in (0, 1)\) with \(\lambda_1 + \lambda_2 + \lambda_3 \geq 2\), there exists an entire solution \(W_j^*(t) = \left\{W_j^*(t)\right\}_{j \in \mathbb{Z}} := \left\{(U_j^*(t), V_j^*(t))\right\}_{j \in \mathbb{Z}}\) of (1.3) such that

\[
\max \{\chi_1\Phi_1(-j + c_1(t + h_1), \chi_2\Phi_2(j + c_2t + h_2), \chi_3\Gamma_j(t + h_3)\} \leq W_j^*(t) \leq \min \{K_j, \Pi_1(j, t), \Pi_2(j, t), \Pi_3(j, t)\} \quad (4.6)
\]

for \((n, t) \in \mathbb{Z} \times \mathbb{R}\), where \(p := p_{h_1, h_2, h_3} = (\chi_1c_1, \chi_2c_2, \chi_3h_1, \chi_2h_2, \chi_3h_3)\).

**Proof.** By Lemmas 4.2 and 4.5, we have

\[
W_j^*(t) \leq W_j^{n-1}(t) \leq W_j^*(t) \leq \min \{1, \Pi_1(j, t), \Pi_2(j, t), \Pi_3(j, t)\}
\]

for any \(j \in \mathbb{Z}\) and \(t \geq -n\). From the priori estimate (Lemma 4.3) and by a diagonal extraction process, there exists a subsequence \(\{W_j^*(t)\}_{k \to \infty}\) of \(W_n^*(t)\) such that \(W_j^*(t)\) converges to a function \(W_j^*(t) = \left\{W_j^*(t)\right\}_{j \in \mathbb{Z}} := \left\{(U_j^*(t), V_j^*(t))\right\}_{j \in \mathbb{Z}}\) in \(T\). In view of \(W_j^*(t) \leq W_j^{n-1}(t)\) for any \(t > -n\), we have \(\lim_{j \to -\infty} W_j^*(t) = W_j^*(t)\) for any \((j, t) \in \mathbb{Z} \times \mathbb{R}\). The limit function is unique, whence all of the functions \(W_j^*(t)\) converge to the function \(W_j^*(t)\) in \(T\) as \(j \to +\infty\). Clearly, \(W_j^*(t)\) is an entire solution of (1.3) satisfying (4.6). The proof is complete. □
4.3. Qualitative properties of entire solutions

In this subsection, we further investigate some qualitative properties of the entire solutions. For any \( N \in \mathbb{Z} \) and \( a \in \mathbb{R} \), let us denote the regions \( T_{N,a}^i \), \( i = 1, \ldots, 6 \), by

\[
T_{N,a}^1 := (\infty, N) \times (a, \infty), \quad T_{N,a}^2 := [N, \infty) \times (a, \infty), \quad T_{N,a}^3 := Z \times [a, \infty), \\
T_{N,a}^4 := [N, \infty) \times (\infty, a), \quad T_{N,a}^5 := (\infty, N] \times (\infty, a), \quad T_{N,a}^6 := Z \times (\infty, a].
\]

In the following theorem, we say that the functions \( W_p(t) = \{(u_p(t), v_p(t))\}_{j \in \mathbb{Z}} \) converge to a function \( W_{p_0}(t) = \{(u_{p_0}(t), v_{p_0}(t))\}_{j \in \mathbb{Z}} \) as \( p \to p_0 \in \mathbb{R}^2 \) in the sense of the topology \( T \) if, for any compact set \( S \subset \mathbb{R} \times \mathbb{R} \), the functions \( u_p(t), v_p(t) \) in \( T \) uniformly and \( \sup \) converge uniformly in \( S \) to \( u_{p_0}(t), v_{p_0}(t) \) and \( \sup \) as \( p \to p_0 \).

**Theorem 4.7.** Let \( W_p(t) = \{(W_j(t), V_j(t))\}_{j \in \mathbb{Z}} \) be the entire solution of (1.3) as stated in Theorem 4.6, then the following properties hold.

(i) \( W_p(t) \to 0 \) and \( W_j(t) \to 0 \) for all \( j \in \mathbb{Z} \) and \( t \in \mathbb{Z} \)

(ii) \( \lim_{t \to \infty} \sup_{j \in \mathbb{Z}} ||W_j(t)| = 0 \) and \( \lim_{t \to \infty} \sup_{j \in \mathbb{Z}} ||W_j(t)| = 0 \) for any \( N \in \mathbb{N} \).

(iii) For any \( t_0 \in \mathbb{R} \), if \( \chi_1 = 1 \), \( \lim_{t \to \infty} \sup_{j \in \mathbb{Z}} ||W_j(t)| = 0 \) and \( \chi_2 = 1 \), \( \lim_{t \to \infty} \sup_{j \in \mathbb{Z}} ||W_j(t)| = 0 \).

(iv) If \( \chi_3 = 0 \), then for every \( j \in \mathbb{Z} \), there exist \( B_j, D_j \in \mathbb{R}^2 \) with \( 0 < B_j < D_j \) such that

\[
B_j e^{c_i |c_i|^2 t} \leq W_j(t) \leq D_j e^{c_i |c_i|^2 t}
\]

for all \( t < -1 \), where \( c_1, c_2, c_3 = \min \{ c_1 \chi_1(c_1), c_2 \chi_2(c_2) \} \).

(v) For any \( j \in \mathbb{Z} \) and \( t \in \mathbb{R} \), \( W_j(t) \) is increasing with respect to \( h_i \), \( i = 1, 2, 3 \).

(vi) For any \( N \in \mathbb{Z} \) and \( a \in \mathbb{R} \), \( W_p(t) \) converges to \( K \) in \( T \) as \( h_i \to +\infty \) and uniformly on \( (j, t) \in T_{N,a}^i \), \( i = 1, 2, 3 \).

**Proof.** Using (4.6), the proofs for (ii)-(vi) are straightforward and omitted.

We now prove (i). Clearly, \( W_p(t) \to 0 \) for all \( j \in \mathbb{Z} \) and \( t \in \mathbb{R} \). According to \( W_p(t) \to 0 \) \( W_j(t) \to 0 \), we have \( \phi^p = \phi^j \to 0 \) for all \( (j, t) \in \mathbb{Z} \setminus \{ (j_0, t_0) \in \mathbb{Z} \times \mathbb{R} \} \) by the order-preserving of the solution semiflow (see Lemma 4.2). This yields \( \frac{\partial}{\partial t} W_p(t) \to 0 \) for all \( (j, t) \in \mathbb{Z} \times \mathbb{R} \). Next, we show that \( \frac{\partial}{\partial t} U_p(t) \to 0 \) and \( \frac{\partial}{\partial t} V_p(t) \to 0 \) for all \( j \in \mathbb{Z} \) and \( t \in \mathbb{R} \). Note that

\[
\begin{align*}
U_j'(t) &= d_{j,1} U_{j,1}(t) + d_{j,2} U_{j,2}(t) - (d_{j,1} + d_j) U_j(t) \\
&\quad + \left(f_j(U_j(t)) - d_j U_j(t) + \beta_j V_j(t) \right), \\
V_j'(t) &= \alpha_j U_j(t) - \beta_j V_j(t),
\end{align*}
\]

for \( j \in \mathbb{Z} \) and \( t \in \mathbb{R} \). For any \( j \in \mathbb{Z} \) and \( t < 0 \),

\[
U_j'(t) = U_j'(t) = U_j'(t) e^{-\mu_j(t-s)} + \int_s^t h_j(s)e^{-\mu_j(t-s)} ds, \quad (4.7)
\]

\[
V_j'(t) = V_j'(t) = V_j'(t) e^{-\beta_j(t-s)} + \alpha_j \int_s^t U_j(s) e^{-\beta_j(t-s)} ds, \quad (4.8)
\]

where \( \mu_j = d_{j,1} + d_j + \alpha_j + \max_{(j,j',i) \in \mathbb{Z} \times \{1,2\}} |f_i'(u_j)| \) and

\[
h_j(t) = d_{j,1} U_{j,1}(t) + d_{j,2} U_{j,2}(t) + \max_{(j,j',i) \in \mathbb{Z} \times \{1,2\}} |f_i'(u_j)| U_j(t) + \beta_j V_j(t).
\]

Clearly, \( h_j(t) \to 0 \) for all \( j \in \mathbb{Z} \) and \( t \in \mathbb{R} \). Suppose on the contrary that there exist \( (j_0, t_0) \in \mathbb{Z} \times \mathbb{R} \) such that \( U_{j_0}(t_0) = 0 \), then it follows from (4.7) that \( U_{j_0}(t) = 0 \) for all \( t \leq t_0 \). Hence \( U_{j_0}(t) = U_{j_0}(t) \) for all \( t \leq t_0 \), which implies that \( \lim_{t \to t_0} U_{j_0}(t) = U_{j_0}(t_0) \). But following from (4.6), \( \lim_{t \to t_0} U_{j_0}(t) = 0 \) and \( U_{j_0}(t) > 0 \). This contradiction yields that \( \frac{\partial}{\partial t} U_p(t) > 0 \) for all \( j \in \mathbb{Z} \) and \( t \in \mathbb{R} \). Similarly, using (4.8), one can show \( \frac{\partial}{\partial t} V_p(t) > 0 \) for all \( j \in \mathbb{Z} \) and \( t \in \mathbb{R} \). This completes the proof. \( \square \)

Moreover, we denote the entire solution \( W_p(t) \) of (1.3) by

\[
W_p(t) := \begin{cases} 
W_p(t) = \{(U_{p_0}(t), V_{p_0}(t))\}_{j \in \mathbb{Z}}, & \text{if } (\chi_1, \chi_2, \chi_3) = (1, 1, 1); \\
W_p(t) = \{(U_{p_1}(t), V_{p_1}(t))\}_{j \in \mathbb{Z}}, & \text{if } (\chi_1, \chi_2, \chi_3) = (0, 1, 1); \\
W_p(t) = \{(U_{p_2}(t), V_{p_2}(t))\}_{j \in \mathbb{Z}}, & \text{if } (\chi_1, \chi_2, \chi_3) = (1, 0, 1); \\
W_p(t) = \{(U_{p_3}(t), V_{p_3}(t))\}_{j \in \mathbb{Z}}, & \text{if } (\chi_1, \chi_2, \chi_3) = (1, 1, 0),
\end{cases}
\]

where

\[
p = (\chi_1 c_1, \chi_2 c_2, \chi_3 c_3, \chi_1 h_1, \chi_2 h_2, \chi_3 h_3), \quad p_0 = (c_1, c_2, h_1, h_2, h_3), \quad p_1 = (0, c_2, 0, h_2, h_3), \quad p_2 = (c_1, 0, h_1, 0, h_3), \quad \text{and} \quad p_3 = (c_1, c_2, h_1, h_2, 0).
\]

Then we have the following convergence results.
Theorem 4.8. From (4.9), we have the following properties.

(i) For any $N \in \mathbb{Z}$ and $a \in \mathbb{R}$, $W_{p_1}(t)$ converges (in the sense of topology $T$) to

$$
\begin{align*}
W_{p_1}(t) & \text{ as } h_1 \to -\infty, \text{ and uniformly on } (j, t) \in T_{N,a}^1; \\
W_{p_2}(t) & \text{ as } h_2 \to -\infty, \text{ and uniformly on } (j, t) \in T_{N,a}^5; \\
W_{p_3}(t) & \text{ as } h_3 \to -\infty, \text{ and uniformly on } (j, t) \in T_{N,a}^6.
\end{align*}
$$

(ii) For any $N \in \mathbb{Z}$ and $a \in \mathbb{R}$, $W_{p_1}(t)$ converges (in the sense of topology $T$) to

$$
\begin{align*}
\Gamma(t + h_1) & \text{ as } h_2 \to -\infty, \text{ and uniformly on } (j, t) \in T_{N,a}^5; \\
\Phi_{c_1}(t) & \text{ as } h_3 \to -\infty, \text{ and uniformly on } (j, t) \in T_{N,a}^6.
\end{align*}
$$

(iii) For any $N \in \mathbb{Z}$ and $a \in \mathbb{R}$, $W_{p_2}(t)$ converges (in the sense of topology $T$) to

$$
\begin{align*}
\Gamma(t + h_1) & \text{ as } h_1 \to -\infty, \text{ and uniformly on } (j, t) \in T_{N,a}^4; \\
\Phi_{c_1}(t) & \text{ as } h_3 \to -\infty, \text{ and uniformly on } (j, t) \in T_{N,a}^6.
\end{align*}
$$

(iv) For any $N \in \mathbb{Z}$ and $a \in \mathbb{R}$, $W_{p_3}(t)$ converges (in the sense of topology $T$) to

$$
\begin{align*}
\Phi_{c_1}(t) & \text{ as } h_1 \to -\infty, \text{ and uniformly on } (j, t) \in T_{N,a}^4; \\
\Phi_{c_1}(t) & \text{ as } h_2 \to -\infty, \text{ and uniformly on } (j, t) \in T_{N,a}^5.
\end{align*}
$$

Proof. (i) We only prove the case that $W_{p_1}(t)$ converges to $W_{p_1}(t)$ in the sense of topology $T$ as $h_1 \to -\infty$, and uniformly on $(j, t) \in T_{N,a}^1$. The proofs for the other cases are similar.

For $(Z_1, Z_2, Z_3) = (1, 1, 1)$, we denote $\phi^n = \{\phi^n_j\}_{j \in \mathbb{Z}}$ by $\phi^n_p = \{\phi^n_j\}_{j \in \mathbb{Z}}$ and $W^n(t) = \{W^n_j(t)\}_{j \in \mathbb{Z}}$ by $W^n_p(t) = \{W^n_j(t)\}_{j \in \mathbb{Z}}$, respectively. Similarly, when $(Z_1, Z_2, Z_3) = (0, 1, 1)$, we denote $\phi^n_p$ by $\phi^n_p$, and $W^n(t)$ by $W^n_p(t)$, respectively. Let

$$
Z^n(t) = \left\{Z^n_j(t) \right\}_{j \in \mathbb{Z}} := \left\{Z^n_j(t), Z^n_{j+1}(t) \right\}_{j \in \mathbb{Z}},
$$

then $0 \leq Z^n_j(t) \leq K$ for all $(j, t) \in \mathbb{Z} \times (-\infty, +\infty)$ and

$$
\begin{align*}
\frac{d}{dt} Z^n_j(t) & = d_{j,1} Z^n_{j+1}(t) + d_{j,2} Z^n_{j-1}(t) - (d_{j,1} + d_{j,2}) Z^n_j(t) \\
& + f_j(t) N_j Z^n_j(t) - \alpha_j Z^n_j(t), \quad j \in \mathbb{Z}, t > -n.
\end{align*}
$$

Define the function

$$
W(t) = \left\{W_j(t) \right\}_{j \in \mathbb{Z}} = \left\{\left(\tilde{W}_j(t), \tilde{W}_{j+1}(t) \right) \right\}_{j \in \mathbb{Z}}
$$

by

$$
\tilde{W}_j(t) := e^{\lambda(t) - j} \left(\phi_{j+1}(\lambda_1(c_1)), \phi_{j+1}(\lambda_1(c_1)) \right), \quad t \in \mathbb{R}.
$$

Noting that $(\phi_{j+1}(\lambda_1), \phi_{j+1}(\lambda_1))_{j \in \mathbb{Z}} \in K_{\per}$ is the eigenvector corresponding to the principle eigenvalue $M(\lambda_1)$ of (2.1) and $M(\lambda_1) = \lambda_1$, we obtain

$$
\begin{align*}
\frac{d}{dt} \tilde{W}_j(t) & = d_{j,1} \tilde{W}_{j+1}(t) + d_{j,2} \tilde{W}_{j-1}(t) - (d_{j,1} + d_{j,2}) \tilde{W}_j(t) \\
& + f_j(t) \tilde{W}_j(t) - \alpha_j \tilde{W}_j(t), \quad j \in \mathbb{Z}, t \in \mathbb{R}.
\end{align*}
$$

Note also that for all $j \in \mathbb{Z}$,

$$
0 \leq Z^n_j(-n) = \phi^n_{p_1} - \phi^n_{p_1} \leq \Phi_{j+1}(h_1) \leq e^{\lambda(t) - j - c_1 n + h_1} \left(\phi_{j+1}(\lambda_1(c_1)), \phi_{j+1}(\lambda_1(c_1)) \right) = \tilde{W}_j(-n).
$$

It then follows from Lemma 4.4 that

$$
0 \leq Z^n_j(t) = W^n_{p_1}(t) - W^n_{p_1}(t) \leq \tilde{W}_j(t) \leq e^{\lambda(t) - j - c_1 n + h_1} \left(\max_{j \in \mathbb{Z}} \phi_{j+1}(\lambda_1(c_1)), \max_{j \in \mathbb{Z}} \phi_{j+1}(\lambda_1(c_1)) \right)
$$

for all $(j, t) \in \mathbb{Z} \times (-n, +\infty)$. Since $\lim_{n \to -\infty} W^n_{p_1}(t) = W_{p_1}(t)$, $i = 1, 2$, we get

$$
0 \leq W_{j, p_1}(t) - W_{j, p_1}(t) \leq e^{\lambda(t) - j - c_1 n + h_1} \left(\max_{j \in \mathbb{Z}} \phi_{j+1}(\lambda_1(c_1)), \max_{j \in \mathbb{Z}} \phi_{j+1}(\lambda_1(c_1)) \right)
$$
Next, we perform numerical simulations on the existence of entire solutions. For simplicity, we consider the case starting from the supersolution for all based on the following finite difference approximation with a forward scheme for the time derivative For the sake of simplicity, we only consider the case of (1.1) (up to extraction of some subsequence) in , which turns out to be . The limit does not depend on the sequence , whence all of the functions converge to in as . The proofs of (ii)–(iv) are similar to that of (i) and thus omitted. This completes the proof. □

5. Numerical simulations

In this section, we carry out some numerical simulations to confirm the theoretical results presented in previous sections. For the sake of simplicity, we only consider the case . Note that in this case and . The parameters and the function are taken as and and . Thus, we have and .

Set . Then . The computational results on the existence of traveling fronts are based on the monotone iteration scheme (2.12), i.e.

starting from the supersolution , where

Next, we perform numerical simulations on the existence of entire solutions. For simplicity, we consider the case and . Set . We use obtained by the monotone iteration scheme (5.1) instead of the traveling wave fronts and , respectively. The computational results on the existence of entire solutions are based on the following finite difference approximation with a forward scheme for the time derivative.
\[
\begin{align*}
\left\{ \begin{array}{ll}
\frac{u_j^{(k)}-u_j^{(k-1)}}{\Delta t} = u_{j+1}(k-1) + u_{j-1}(k-1) - 2u_j(k-1) - u_j^2(k-1) + v_j(k-1), \\
\frac{v_j^{(k)}-v_j^{(k-1)}}{\Delta t} = u_j(k-1) - v_j(k-1), \\
(u_j(-20), v_j(-20)) = \max \left\{ \phi^{50}_{G1}(j-20c_1), \phi^{50}_{G1}(j-20c_2) \right\},
\end{array} \right.
\end{align*}
\]

where \(\Delta t\) denotes the step size in time and is chosen as \(\Delta t = 0.01\).

From Figs. 1 and 2, the numerical results demonstrate the existence of traveling wave fronts and entire solutions.

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References


