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Travelling waves in delayed reaction–diffusion equations on higher dimensional lattices
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This paper is concerned with travelling wave solutions for a class of monostable delayed reaction–diffusion equations on higher dimensional lattices. We first show that, for any fixed unit vector $\sigma \in \mathbb{R}^n$, there is a minimal wave speed $c_*(\sigma)$ such that a travelling front exists if and only if its speed is above this minimal speed. The exact asymptotic behaviour of the wave profiles at infinity is then established. Finally, we show that any travelling wave solution is strictly monotone and unique (up to a translation), including even the minimal wave. Of particular interest is the effects of the delay, spatial dimension and direction of wave on the minimal wave speed and we obtain some interesting phenomena for the delayed lattice dynamical system which are different from the case when the spatial variable is continuous.

Keywords: travelling wave solution; discrete reaction–diffusion equation; monostable nonlinearity; quasi-monotone condition

AMS Subjective Classifications (2000): 34K05; 35K57; 34B40; 34D23; 34K60

1. Introduction

This paper is concerned with the travelling wave solutions of the $n$-dimensional spatially discrete reaction–diffusion equation with delay and monostable nonlinearity [26]

$$u'_{\eta}(t) = D(\Delta_n u)_{\eta} + f(u_{\eta}(t), u_{\eta}(t - \tau)), \quad (1.1)$$

where $n \in \mathbb{Z}_+, \eta \in \mathbb{Z}^n$, $t > 0$, $u_{\eta}(t) \in \mathbb{R}$, $D > 0$, $\tau \geq 0$ are constants, $(\Delta_n u)_{\eta}$ is the standard $n$-dimensional discrete Laplacian, i.e.

$$(\Delta_n u)_{\eta} = \sum_{\|\eta - \eta'\|_1 = 1, \eta' \in \mathbb{Z}^n} [u_{\eta'}(t) - u_{\eta}(t)]$$

and the reaction function $f$ satisfying

(A1) $f \in C^2([0, K]^2, \mathbb{R})$, $f(0, 0) = f(K, K) = 0$, $f(u, u) > 0$ for all $u \in (0, K)$, and $
\partial_1 f(K, K) + \partial_2 f(K, K) < 0$, where $K > 0$ is a constant;

(A2) $\partial_2 f(u, v) \geq 0$ and $f(u, v) \leq \partial_1 f(0, 0)u + \partial_2 f(0, 0)v$ for all $(u, v) \in [0, K]^2$.

From (A1) and (A2), we can see that (1.1) has two equilibria 0 and $K$, and $\partial_1 f(0, 0) + \partial_2 f(0, 0) \geq 2f(K/2, K/2)/K > 0$. Clearly, (A1) and (A2) are standard

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monostable assumption, quasi-monotone and sub-tangential conditions. We are interested in travelling wave solutions of (1.1) that connect the two equilibria 0 and K. Throughout this paper, a travelling wave solution always refers to a trinity \( (U, c, \sigma) \), where \( U = U(\cdot) : \mathbb{R} \to [0, K] \) is a function, \( c > 0 \) is a constant and \( \sigma \in \mathbb{R}^n \) is a unit vector, such that \( u_\eta(t) := U(\eta \sigma + ct) \) is a solution of (1.1), and
\[
U(-\infty) := \lim_{\xi \to -\infty} U(\xi) = 0, \quad U(+\infty) := \lim_{\xi \to +\infty} U(\xi) = K. \tag{1.2}
\]

The vector \( \sigma \) represents the direction of the wave. We call \( c \) the wave speed and \( U \) the wave profile. Moreover, we say \( U \) is a travelling (wave) front if \( U(\cdot) : \mathbb{R} \to \mathbb{R} \) is monotone.

Lattice differential equations occur in many different applied fields, such as biology [7,19] and material science [1]. In addition, they can also be viewed as the results of discretizing the corresponding models of spatially continuous partial differential equations. For example, equation (1.1) is a spatial discretization of the reaction–diffusion equation with delay
\[
u_t(x,t) = D \Delta u(x,t) + f(u(x,t), u(x, t - \tau)), \quad x \in \mathbb{R}^n, \quad t > 0, \tag{1.3}
\]
where \( \Delta \) is the usual Laplacian operator on \( \mathbb{R}^n \). It is interesting and worthwhile to compare the dynamics of (1.1) with that of (1.3). It is known that the diffusion (Laplacian) term in the reaction–diffusion equation, such as (1.3), is from Fick’s law, that is to say, all the individuals carry out an unbiased random walk. However, for the spatially discrete equation, the diffusion interaction occurs through spatial dispersal only among adjacent finite points of the lattice. So, as mentioned by Zou [26], an anisotropy in directional dependence is often introduced in discretizing the \( n \)-dimensional Laplacian for \( n \geq 2 \), and thus, spatially discrete equations often exhibit more complicated and richer dynamics than spatially continuous equations, see, e.g. [2,8,9,17,18,26].

Travelling wave solutions are an important class of solutions for a large class of nonlinear dynamical systems describing a variety of physical and biological phenomena. In the past decades, the travelling waves of lattice differential equations have been studied extensively and intensively. Many interesting and important results have been established, some of which have revealed some essential difference between a discrete model and its continuous version, see, e.g. [3–6,10,13,18,24,25], and the surveys of [8,9].

It is well known that time delay should be and has been incorporated into many realistic models in applications. Mathematically, significant additional technical difficulties arise from the presence of time delay in the study of travelling wave solutions. In fact, the presence of delay in an ODE changes a finite-dimensional system to an infinite-dimensional system [16]. The existence of travelling fronts of delayed lattice differential equations was initially studied by Wu and Zou [22] and Zou [26]. In [22], Wu and Zou studied the travelling fronts of (1.1) with \( n = 1 \) and general delay. Zou [26] further considered the travelling fronts of (1.1) with general dimension \( n \). More precisely, he reduced the existence of the travelling fronts to that of an admissible pair of supersolution and subsolution by establishing a monotone iteration starting from a supersolution. However, they did not consider the non-existence, monotonicity and uniqueness of travelling waves. The goal of this paper is to resolve these issues. For other related results on the travelling waves of delayed lattice differential equations, we refer to [7,11,12,14,16,19] and the references therein.

In this paper, we study systematically the properties of the travelling waves of (1.1), including the existence, non-existence, asymptotic behaviour, monotonicity, uniqueness,
symmetry and periodicity. We first show that, for any fixed unit vector \( \sigma \in \mathbb{R}^n \), there exists a positive minimal wave speed \( c_*(\sigma) \) (depending on \( \tau \) and \( n \)) such that (1.1) has no travelling wave for \( 0 < c < c_*(\sigma) \) and has a travelling front \( U(\xi) := (U(\xi), c, \sigma) \) for each \( c \geq c_*(\sigma) \) (Theorem 2.4). The effects of the delay \( \tau \), the spatial dimension \( n \) and the direction \( \sigma \) on the minimal wave speed are then considered. We find that the dimension \( n \) will slow the diagonal waves and does not affect the speed of the \( 1-D \) waves, and that (i) if \( \partial_2 f(0, 0) > 0 \), then the delay will slow the minimal wave speed, and (ii) if \( \partial_2 f(0, 0) = 0 \), then the delay does not affect the minimal wave speed (Theorems 2.6 and 2.8). This is probably the first time the effect of the spatial dimension of lattice dynamical systems on the minimal wave speed has been studied. For \( n = 2 \) and \( \sigma = (\cos \theta, \sin \theta) \), \( \theta \in \mathbb{R} \), we show that (i) \( c_*(\theta) \) is a periodic function of period \( \pi/2 \) and has the symmetry in \( \theta = \pi/4 \); (ii) \( c_*(\theta) \) is maximal for \( \theta = 0 \) and \( \theta = \pi/2 \) and minimal for \( \theta = \pi/4 \) and (iii) \( c_*(\theta) \) is decreasing in \( \theta \in [0, \pi/4] \) and increasing in \( \theta \in [\pi/4, \pi/2] \) (Theorem 2.6). This finding implies that the \( 1-D \) wave (east/west or north/south) is the fastest and the diagonal wave is the slowest.

Furthermore, the exact asymptotic behaviour of the wave profiles as \( \xi \to \pm \infty \) is established by using the famous Ikehara’s theorem [3]. Based on the asymptotic behaviour of the wave profiles, we then show that any travelling wave with given direction \( \sigma \) and speed \( c \geq c_*(\sigma) \) is strictly monotone increasing and unique (up to a translation), including even the minimal wave (Theorems 4.2 and 4.5). Our proof relies on the comparison principle and a sliding method developed by Chen and Guo [5] for a \( 1-D \) lattice differential equation without delay. As a by-product, we obtain the periodicity and symmetry of the travelling waves for the case \( n = 2 \). More precisely, we show that, for any \( c \geq c_*(\theta) \), \( U_{c, \theta}(\cdot) := (U(\cdot), c, \theta) \), \( \theta \in \mathbb{R} \) is a periodic function of period \( \pi/2 \) with respect to \( \theta \) and has the symmetry in \( \theta = \pi/4 \) with phase shift (Theorem 4.6).

The rest of this paper is organized as follows. In Section 2, we first prove the existence of the minimal wave speed. Some properties of the travelling waves are also given. Then, we consider the effects of the delay \( \tau \), the spatial dimension \( n \) and the direction \( \sigma \) on the minimal wave speed. Section 3 is devoted to the exact asymptotic behaviour of the wave profiles. In Section 4, we prove the monotonicity, uniqueness, periodicity and symmetry of the travelling waves. Finally, in Section 5, we apply our results to two specific biological models and obtain some new results which improve and/or complement some existing results in [16,26].

2. Properties of minimal wave speed

In the remainder of this paper we always assume that (A1) and (A2) hold, and take \( L_i = \max_{(u,v) \in [0,K]^2} |\partial_i f(u,v)| \), \( i = 1, 2 \), and \( L = \max_{i,j=1,2} \left\{ \max_{(u,v) \in [0,K]^2} |\partial_i \partial_j f(u,v)| \right\} \).

2.1 Existence of minimal wave speed

We remark that the existence of the travelling fronts is obtained using the standard monotone iteration approach and limiting argument as in [16,24,26]. So, we only sketch the outline.

Substituting \( U(\xi) \), \( \xi = \eta \cdot \sigma + ct \) into (1.1) we obtain the corresponding wave equation

\[
cU'(\xi) = E[U](\xi) + f(U(\xi), U(\xi - c\tau)),
\] (2.1)
where
\[
E[U](\xi) := D \sum_{k=1}^{n} [U(\xi + \sigma_k) + U(\xi - \sigma_k) - 2U(\xi)].
\]  
(2.2)

For \(c \geq 0\) and \(\lambda \in \mathbb{C}\), we define two functions:
\[
\Delta_0(c, \lambda) = c\lambda - D \sum_{k=1}^{n} [e^{\lambda \sigma_k} + e^{-\lambda \sigma_k} - 2] - \partial_1 f(0, 0) - \partial_2 f(0, 0)e^{-\lambda c \tau} \quad \text{and}
\]
\[
\Delta_1(c, \lambda) = c\lambda - D \sum_{k=1}^{n} [e^{\lambda \sigma_k} + e^{-\lambda \sigma_k} - 2] - \partial_1 f(K, K) - \partial_2 f(K, K)e^{-\lambda c \tau}.
\]

**Lemma 2.1.** For any fixed \(\sigma \in \mathbb{R}^n\) with \(|\sigma| = 1\), the following results hold:

(i) There exists a number \(c_+(\sigma) > 0\) (depending on \(\tau\) and \(n\)) such that

(a) if \(0 < c < c_+(\sigma)\) and \(\lambda \geq 0\), then \(\Delta_0(c, \lambda) < 0\);

(b) if \(c \geq c_+(\sigma)\), then the equation \(\Delta_0(c, \lambda) = 0\) has two positive real roots \(\lambda_1(c)\) and \(\lambda_2(c)\) with \(\lambda_1(c) \leq \lambda_2(c)\);

(c) if \(c = c_+(\sigma)\), then \(\lambda_1(c) = \lambda_2(c) := \lambda_*\), and if \(c > c_+(\sigma)\), then \(\lambda_1(c) < \lambda_* < \lambda_2(c)\), \(\lambda'_1(c) < 0\), \(\lambda'_2(c) > 0\), and
\[
\Delta_0(c, \lambda) = \begin{cases} < 0, & \text{for } \lambda \in \mathbb{R} \setminus (\lambda_1(c), \lambda_2(c)), \\ > 0, & \text{for } \lambda \in (\lambda_1(c), \lambda_2(c)). \end{cases}
\]

(ii) The equation \(\Delta_1(c, \lambda) = 0\) has two real roots \(\lambda_3(c) < 0\) and \(\lambda_4(c) > 0\).

**Definition 2.2.** For any fixed \(\sigma \in \mathbb{R}^n\) with \(|\sigma| = 1\), an absolutely continuous function \(\phi : \mathbb{R} \rightarrow \mathbb{R}\) is called a supersolution (subsolution) of (2.1) if \(\phi\) satisfies
\[
H_c[\phi](\xi) := c \lim_{h \to 0^+} \frac{\phi(\xi + h) - \phi(\xi)}{h} - E[U](\xi) - f(U(\xi), U(\xi - c\tau)) \geq (\leq) 0,
\]
a.e. on \(\mathbb{R}\).

**Lemma 2.3.** For any fixed \(\sigma \in \mathbb{R}^n\) with \(|\sigma| = 1\), let \(c > c_+(\sigma)\) and \(\lambda_1(c), \lambda_2(c)\) be defined as in Lemma 2.1. Then, for every \(\nu \in (1, \min \{2, \lambda_2(c)/\lambda_1(c)\})\), there exists \(Q(c, \nu) \geq 1\), such that for any \(l \geq Q(c, \nu)\), the functions \(\phi^\pm\) defined by
\[
\phi^+(\xi) = \min \{K, e^{\lambda_1(c)\xi} + le^{\lambda_2(c)\xi}\}, \quad \phi^-(\xi) = \max \{0, e^{\lambda_1(c)\xi} - le^{\lambda_2(c)\xi}\}, \quad \xi \in \mathbb{R},
\]
are a supersolution and a subsolution of (2.1), respectively.

**Theorem 2.4.** For any fixed \(\sigma \in \mathbb{R}^n\) with \(|\sigma| = 1\), the following results hold:

(i) For every \(c \geq c_+(\sigma)\), (1.1) has a travelling front \((U(\cdot), c, \sigma)\).

(ii) For \(0 < c < c_+(\sigma)\), (1.1) has no travelling wave solutions.
Proof. Noting that $\partial_2 f(u, v) \geq 0$ for $(u, v) \in [0, K]^2$, the existence of the travelling fronts when $c > c_*(\sigma)$ is a direct corollary of Lemma 2.3 and [26, Theorem 2.5].

For the case $c = c_*(\sigma)$, the existence of the travelling front could be obtained by a limiting argument similar to that of [23, Theorem 3.1].

The non-existence of travelling waves when $0 < c < c_*(\sigma)$ will be proved in Section 3. This completes the proof. □

We complete this subsection with a result on some properties of travelling waves.

Lemma 2.5. For any fixed $\sigma \in \mathbb{R}^n$ with $|\sigma| = 1$, let $(U, c, \sigma)$ be a travelling wave solution of (1.1) with direction $\sigma$ and speed $c \geq c_*(\sigma)$. Then,

(i) $U$ satisfies $0 < U(\cdot) < K$ on $\mathbb{R}$;

(ii) if $U'(\cdot) \geq 0$ on $\mathbb{R}$, then $U'(\cdot) > 0$ on $\mathbb{R}$.

Proof. By the symmetry of $E[U](\xi)$, we assume that $\sigma_k \geq 0, k = 1, \cdots, n$, i.e. $\sigma \in \mathbb{R}_+^n$. Let $\sigma_{k_0} = \max_{k=1, \cdots, n} \{\sigma_k\}, k_0 \in \{1, \cdots, n\}$. Then $\sigma_{k_0} > 0$, since $|\sigma| = 1$.

(i) Suppose on the contrary that there exists $\xi_0 \in \mathbb{R}$ such that $U(\xi_0) = 0$. Since $U(+\infty) = K$ and $0 \leq U(\cdot) \leq K$, $\xi_0 := \sup\{\xi \in \mathbb{R} | U(\xi) = 0\}$ is well defined and $U(\xi_0) = U'(\xi_0) = 0$. Thus,

$$0 = cU'(\xi_0) = D \sum_{k=1}^n [U(\xi_0 + \sigma_k) + U(\xi_0 - \sigma_k)] + f(0, U(\xi_0) + c\tau) \geq DU(\xi_0 + \sigma_{k_0}) \geq 0,$$

it follows that $U(\xi_0 + \sigma_{k_0}) = 0$. This contradicts the definition of $\xi_0$. Hence, $U(\cdot) > 0$ on $\mathbb{R}$. Similarly, we can prove that $U(\cdot) < K$ on $\mathbb{R}$.

(ii) Let $\mu = 2nD + L_1$, and define

$$H[U](\xi) = D \sum_{k=1}^n [U(\xi + \sigma_k) + U(\xi - \sigma_k)] + L_1 U(\xi) + f(U(\xi), U(\xi + c\tau)).$$

Since $U'(\xi) \geq 0$ for $\xi \in \mathbb{R}$ and $\partial_2 f(u, v) \geq 0$ for $(u, v) \in [0, K]^2$, one can show that $H[U](\xi)$ is also non-decreasing in $\xi \in \mathbb{R}$. Moreover, $U$ satisfies the equation

$$cU(\xi) = e^{-\mu \xi} \int_{-\infty}^{\xi} e^{\mu s} H[U](s) ds.$$

Suppose that there exists $\xi_0 \in \mathbb{R}$ such that $U'(\xi_0) = 0$. Then,

$$0 = cU'(\xi_0) = -\frac{\mu}{c} e^{-\mu \xi_0} \int_{-\infty}^{\xi_0} e^{\mu s} \{H[U](s) - H[U](\xi_0)\} ds.$$

Noting that $H[U](s) \leq H[U](\xi_0)$ for all $s \leq \xi_0$. Thus, $H[U](s) = H[U](\xi_0)$ for all $s \leq \xi_0$. Letting $s \to -\infty$, we obtain $H[U](\xi_0) = 0$. From $cU'(\xi_0) + \mu U(\xi_0) = H[U](\xi_0)$, we get $U(\xi_0) = 0$, which is a contradiction. Hence, $U'(\cdot) > 0$ on $\mathbb{R}$.

2.2 The effects of delay, dimension and direction on $c_*(\sigma)$

In this subsection, we consider the effects of the delay $\tau$, the dimension $n$ and the direction $\sigma$ in (2.1) on the minimal wave speed $c_*(\sigma)$. One shall see that our results reveal some
essential difference between the spatial discrete equation (1.1) and its continuous version (1.3).

It is well known that the minimal wave speed for the continuous reaction–diffusion equations (1.3) is independent of the dimension \(n\) and the direction of wave \(\sigma\). In fact, the wave equation of (1.3) is

\[
    c\phi_t(\xi) = D\phi''(\xi) + f(\phi(\xi), \phi(\xi - c\tau)), \quad \xi \in \mathbb{R},
\]

and the corresponding characteristic equation takes the form

\[
    c\lambda - D\lambda^2 - \partial f(0, 0) - \partial_2 f(0, 0)e^{-\lambda \tau} = 0.
\]

However, as mentioned by Zou [26], the minimal wave speed of the discrete equation (1.1) depends on not only the delay \(\tau\) but also the dimension \(n\) and direction \(\sigma\).

We have the following results.

**Theorem 2.6**

(i) If \(\partial f(0, 0) > 0\), then the delay will slow the minimal wave speed, and if \(\partial f(0, 0) = 0\), then the delay does not affect the minimal wave speed.

(ii) If \(n = 2\) and \(\sigma = (\cos \theta, \sin \theta), \theta \in \mathbb{R}\), then \(c_*(\theta + \pi/2) = c_*(\theta)\) and \(c_*(\pi/2 - \theta)\) for \(\theta \in \mathbb{R}\). Moreover, \(c_*(\theta)\) is monotonically decreasing in \(\theta \in [0, \pi/4]\) and monotonically increasing in \(\theta \in [\pi/4, \pi/2]\), and

\[
    c_*(\frac{\pi}{4}) = \min_{\theta \in [0, \frac{\pi}{4}]} \quad \{c_*(\theta)\}, \quad c_*(0) = c_*\left(\frac{\pi}{2}\right) = \max_{\theta \in [0, \frac{\pi}{2}]} \quad \{c_*(\theta)\}.
\]

**Proof.** From Lemma 2.1, for any \(\sigma \in \mathbb{R}^d\), \((c_*, \lambda_*) := (c_*(\sigma), \lambda_*(\sigma))\) is uniquely determined as the solution of the system

\[
    \Delta_0(c_*, \lambda_*) = 0 \quad \text{and} \quad \frac{\partial}{\partial \lambda} \Delta_0(c_*, \lambda) \bigg|_{\lambda = \lambda_*} = 0. \quad (2.3)
\]

(i) Clearly, \(c_* = c_*(\sigma)\) and \(\lambda_* = \lambda_*(\sigma)\) are differentiable functions with respect to \(\tau\). Furthermore,

\[
    \frac{\partial c_*}{\partial \tau} = -\frac{\partial \Delta_0}{\partial \tau} \bigg|_{(c_*, \lambda_*)} = -\left. \frac{c_* \partial_2 f(0, 0)e^{-\lambda_* \tau}}{1 + \tau \partial_2 f(0, 0)e^{-\lambda_* \tau}} \right|_{(c_*, \lambda_*)} \leq 0,
\]

which implies that if \(\partial f(0, 0) > 0\), then the delay will slow the minimal wave speed, and if \(\partial f(0, 0) = 0\), then the delay does not affect the minimal wave speed.

(ii) For \(n = 2\) and \(\sigma = (\cos \theta, \sin \theta), \theta \in \mathbb{R}\), (2.1) reduces to

\[
    c U'(\xi) = E[U](\xi) + f(U(\xi), U(\xi - c\tau)), \quad (2.4)
\]

where

\[
    E[U](\xi) = D[U(\xi + \cos \theta) + U(\xi - \cos \theta) + U(\xi + \sin \theta) + U(\xi - \sin \theta) - 4U(\xi)]. \quad (2.5)
\]
In this case, \((c_*, \lambda_*) = (c_*(\theta), \lambda_*(\theta))\) is the unique solution of the system

\[
0 = c_* \lambda_* - D \left[ e^{\lambda_* \cos \theta} + e^{-\lambda_* \cos \theta} + e^{\lambda_* \sin \theta} + e^{-\lambda_* \sin \theta} - 4 \right] 
- \partial_1 f(0,0) - \partial_2 f(0,0)e^{-\lambda_* \cos \tau},
\]

\[
0 = c_* - D \left[ e^{\lambda_* \cos \theta} \cos \theta - e^{-\lambda_* \cos \theta} \cos \theta + e^{\lambda_* \sin \theta} \sin \theta - e^{-\lambda_* \sin \theta} \sin \theta \right] 
+ c_* \tau \partial_2 f(0,0)e^{-\lambda_* \cos \tau}.
\]

(2.6)

(2.7)

We first prove \(c_*(\theta) = c_*(\pi/2 - \theta)\) for \(\theta \in \mathbb{R}\). From Lemma 2.1, \((c_1^*, \lambda_1^*) = (c_*(\pi/2 - \theta), \lambda_*(\pi/2 - \theta))\) satisfies

\[
0 = c_1^* \lambda_1^* - D \left[ e^{\lambda_1^* \cos (\pi - \theta)} + e^{-\lambda_1^* \cos (\pi - \theta)} + e^{\lambda_1^* \sin (\pi - \theta)} + e^{-\lambda_1^* \sin (\pi - \theta)} - 4 \right] 
- \partial_1 f(0,0) - \partial_2 f(0,0)e^{-\lambda_1^* \cos \tau},
\]

\[
0 = c_1^* - D \left[ e^{\lambda_1^* \cos (\pi - \theta)} \cos (\pi - \theta) - e^{-\lambda_1^* \cos (\pi - \theta)} \cos (\pi - \theta) + e^{\lambda_1^* \sin (\pi - \theta)} \sin (\pi - \theta) \right] 
\times \sin (\pi - \theta) - e^{-\lambda_1^* \sin (\pi - \theta)} \sin (\pi - \theta) \right]
+ c_1^* \tau \partial_2 f(0,0)e^{-\lambda_1^* \cos \tau},
\]

that is

\[
0 = c_1^* \lambda_1^* - D \left[ e^{\lambda_1^* \cos \theta} + e^{-\lambda_1^* \cos \theta} + e^{\lambda_1^* \sin \theta} + e^{-\lambda_1^* \sin \theta} - 4 \right] 
- \partial_1 f(0,0) - \partial_2 f(0,0)e^{-\lambda_1^* \cos \tau},
\]

\[
0 = c_1^* - D \left[ e^{\lambda_1^* \cos \theta} \cos \theta - e^{-\lambda_1^* \cos \theta} \cos \theta + e^{\lambda_1^* \sin \theta} \sin \theta - e^{-\lambda_1^* \sin \theta} \sin \theta \right] 
+ c_1^* \tau \partial_2 f(0,0)e^{-\lambda_1^* \cos \tau}.
\]

By the uniqueness of solutions of (2.6) and (2.7), we obtain \(c_*(\theta) = c_*(\pi/2 - \theta)\) for \(\theta \in \mathbb{R}\). Similarly, one can show that \(c_*(\theta + \pi/2) = c_*(\theta)\) for \(\theta \in \mathbb{R}\).

Furthermore, by (2.6) and (2.7), we have

\[
\frac{\partial c_*}{\partial \theta} = - \frac{\partial \Delta_0}{\partial \theta} \bigg|_{(c_*, \lambda_*)} 
= \frac{D \left[ e^{\lambda_* \sin \theta} \cos \theta - e^{-\lambda_* \sin \theta} \cos \theta - e^{\lambda_* \cos \theta} \sin \theta + e^{-\lambda_* \cos \theta} \sin \theta \right]}{1 + \tau \partial_2 f(0,0)e^{-\lambda_* \cos \tau}}.
\]

Consider the function

\[
G(\theta) = e^{\lambda_* \sin \theta} \cos \theta - e^{-\lambda_* \sin \theta} \cos \theta - e^{\lambda_* \cos \theta} \sin \theta + e^{-\lambda_* \cos \theta} \sin \theta, \quad \theta \in \left[ 0, \frac{\pi}{2} \right],
\]
we get
\[
G(\theta) = \sum_{i=0}^{\infty} \left\{ \frac{(-\lambda \sin \theta) \cos \theta - (\lambda \cos \theta) \sin \theta}{i!} \right\} + \frac{(-\lambda \cos \theta) \sin \theta + (\lambda \cos \theta) \sin \theta}{i!} = \sum_{k=0}^{\infty} \frac{\lambda^{2k+1}[\sin^2 \theta - \cos^2 \theta] \sin 2\theta}{(2k+1)!}.
\]
Thus, \( G(0) = G(\pi/4) = G(\pi/2) = 0 \), \( G(\theta) \leq 0 \) for \( \theta \in [0, \pi/4] \) and \( G(\theta) \geq 0 \) for \( \theta \in [\pi/4, \pi/2] \), it follows that assertion (ii) holds. This completes the proof. \( \square \)

**Remark 2.7.** From Theorem 2.6 (ii), we see that \( c_\ast(\theta) \) is a periodic function of period \( \pi/2 \) and has the symmetry in \( \theta = \pi/4 \), and \( c_\ast(\theta) \) is maximal for \( \theta = 0 \) and \( \theta = \pi/2 \) and minimal for \( \theta = \pi/4 \). This finding agrees with physical intuition: the \( 1-D \) wave (east/west or north/south) is the fastest and the diagonal wave is the slowest. We point out that a similar phenomenon has been observed by Cheng et al. [7] for a delayed population model on a \( 2-D \) spatial lattice by numerical simulations.

**Theorem 2.8.** For any fixed \( \sigma \in \mathbb{R}^n \) with \( |\sigma| = 1 \), the following holds:

(i) If \( \sigma_1 = \cdots = \sigma_n \), i.e. \( \sigma_i = 1/\sqrt{n}, i = 1, \cdots, n \), the dimension \( n \) will slow the minimal wave speed.

(ii) If there exists \( i_0 \in \{1, \cdots, n\} \) such that \( \sigma_{i_0} = 1 \), then the minimal wave speed is independent of the spatial dimension \( n \).

(iii) \( c_\ast(\sigma_{n+1}) = c_\ast(\sigma), \) where \( \sigma_{n+1} = (\sigma, 0) \).

**Proof.**

(i) For \( \sigma_i = 1/\sqrt{n}, i = 1, \cdots, n \), it is easy to verify that
\[
\Delta_0(c_\ast, \lambda_\ast) = c_\ast \lambda_\ast - Dn \left[ e^{\lambda_\ast \frac{1}{\sqrt{n}}} + e^{-\lambda_\ast \frac{1}{\sqrt{n}}} - 2 \right] - \partial_1 f(0, 0) - \partial_2 f(0, 0) e^{-\lambda_\ast c_\ast \tau}
\]
\[
= c_\ast \lambda_\ast - 2D \sum_{k=1}^{\infty} \frac{\lambda_\ast^{2k}}{(2k)!} n^{k-1} - \partial_1 f(0, 0) - \partial_2 f(0, 0) e^{-\lambda_\ast c_\ast \tau}.
\]
Thus, we have
\[
\frac{\partial c_\ast}{\partial n} = -\frac{\partial \Delta_0}{\partial n} \bigg|_{(c_\ast, \lambda_\ast)} = -2D \sum_{k=1}^{\infty} \frac{\lambda_\ast^{2k}(k-1)}{(2k)!} \frac{1}{n^k} < 0,
\]
which implies that the dimension \( n \) will slow the minimal wave speed of diagonal waves.

(ii) Without loss of generality, we suppose that \( \sigma_1 = 1 \). Since \( |\sigma| = 1 \), \( \sigma = (1, 0, \cdots, 0) \). Then \( (c_\ast(\sigma), \lambda_\ast(\sigma)) \) satisfies
\[
c\lambda - D[e^\lambda + e^{-\lambda} - 2] - \partial_1 f(0, 0) - \partial_2 f(0, 0) e^{-\lambda c_\ast \tau} = 0.
\]
It is easy to see that for \( \sigma = (1, 0, \cdots, 0) \), \( c_\ast(\sigma) \) is independent of the dimension \( n \). Assertion (iii) is direct. This completes the proof. \( \square \)
Remark 2.9. Theorem 2.8 implies that the spatial dimension \( n \) will slow the diagonal waves and does not affect the speed of the \( 1 - D \) waves. This is probably the first time the effect of the spatial dimension of lattice dynamical systems on the minimal wave speed has been studied.

Let \( \sigma^0_{n+1} = (\sigma \cos \theta, \sin \theta) \), \( \theta \in [0, \pi/2) \), Theorem 2.8 (iii) implies that 
\[
    c_*(\sigma^0_{n+1}) = c_*(\sigma).
\]
We conjecture that \( c_*(\sigma^0_{n+1}) < c_*(\sigma) \) for \( \theta \in (0, \pi/2) \).

3. Asymptotic behaviour of travelling waves

To obtain the asymptotic behaviour of the wave profiles, we first provide a technical lemma, which can be found in Carr and Chmaj [3].

Lemma 3.1. Let \( F(\Lambda) := \int_0^{+\infty} e^{-\Lambda \xi} u(\xi) d\xi \) and \( u(\xi) \) be a positive decreasing function. If \( F \)
can be written as \( F(\Lambda) = J(\Lambda + \Lambda_0)^{-(k+1)} \) for some constants \( k > -1, \Lambda_0 > 0 \), and some
analytic function \( J \) in the strip \(-\Lambda_0 \leq \text{Re} \Lambda < 0 \), then \( \lim_{\xi \to +\infty} u(\xi)/(\xi^k e^{-\Lambda_0 \xi}) = J(-\Lambda_0)/\Gamma(\Lambda_0 + 1) \).

In what follows, we denote \( c_* = c_*(\theta) \) for simplicity.

Theorem 3.2. For any fixed \( \sigma \in \mathbb{R}^n \) with \( |\sigma| = 1 \), let \((U, c, \sigma)\) be a travelling wave solution of (1.1) with direction \( \sigma \) and speed \( c \geq c_* \). Then,

(i) for \( c > c_* \),
\[
    \lim_{\xi \to -\infty} U(\xi)e^{-\lambda_1(c)\xi} = a_0(c), \quad \lim_{\xi \to -\infty} U'(\xi)e^{-\lambda_1(c)\xi} = a_0(c)\lambda_1(c) \quad (3.1)
\]

and for \( c = c_* \),
\[
    \lim_{\xi \to -\infty} U(\xi)e^{-\lambda_1(c)\xi} = a_0(c_*), \quad \lim_{\xi \to -\infty} U'(\xi)e^{-\lambda_1(c)\xi} = a_0(c_*)\lambda_1(c_*) \quad (3.2)
\]

(ii) for \( c \geq c_* \),
\[
    \lim_{\xi \to +\infty} [K - U(\xi)]e^{-\lambda_3(c)\xi} = a_1(c), \quad \lim_{\xi \to +\infty} U'(\xi)e^{-\lambda_3(c)\xi} = -a_1(c)\lambda_3(c), \quad (3.3)
\]

where \( a_0(c), a_0(c_*), a_1(c) \) are positive constants.

Proof. We only prove (i), since the proof of assertion (ii) is similar. The proof of (i) is divided into three steps.

Step 1. We show that \( U(\xi) \) is integrable on \((-\infty, \xi')\) for some \( \xi' \in \mathbb{R} \). Denote \( \rho_1 = \frac{\partial f(0,0)}{\partial x}(0,0) + \frac{\partial f(0,0)}{\partial y}(0,0) \) and \( \rho_2 = -\frac{\partial f(0,0)}{\partial x}(0,0) - \frac{\partial f(0,0)}{\partial y}(0,0) \). Note that \( \rho_1 > 0 \) and \( U(-\infty) = 0 \), there exists \( \xi' < 0 \) such that for \( \xi \leq \xi' \),
\[
    L[U^2(\xi) + 2U(\xi)U(\xi - c\tau) + U^2(\xi - c\tau)] \leq \frac{\rho_1}{4} U(\xi) + \frac{\rho_1}{4} U(\xi - c\tau).
\]
Thus, by Taylor’s expansion, for $\xi \leq \xi'$,

$$f(U(\xi), U(\xi - c\tau)) \approx \partial_1 f(0, 0)U(\xi) + \partial_2 f(0, 0)U(\xi - c\tau) - \frac{\partial_1}{4} U(\xi) - \frac{\partial_1}{4} U(\xi - c\tau)$$

$$\geq \frac{\partial_1}{4} U(\xi) + \frac{\partial_2}{2} [U(\xi - c\tau) - U(\xi)],$$

it then follows from (2.1) that for $\xi \leq \xi'$,

$$cU'(\xi) \geq E[U](\xi) + \frac{\partial_1}{4} U(\xi) + \frac{\partial_2}{2} [U(\xi - c\tau) - U(\xi)].$$

(3.4)

Integrating (3.4) over $[y, \xi], y < \xi \leq \xi'$, we obtain

$$c[U(\xi) - U(y)] \geq \int_y^\xi E[U](s)ds + \frac{\partial_1}{4} \int_y^\xi U(s)ds + \frac{\partial_2}{2} \int_y^\xi [U(s - c\tau) - U(s)]ds.$$  

(3.5)

Set

$$Q(x) = D \sum_{k=1}^n \left[ \int_{x - \alpha_k}^{x + \alpha_k} U(s)ds - \int_{x - \alpha_k}^x U(s)ds \right].$$

Thus, $\int_y^\xi E[U](s)ds = Q(\xi) - Q(y)$ and $Q(y) \to 0, y \to -\infty$, which imply that

$$\lim_{y \to -\infty} \int_y^\xi E[U](s)ds = Q(\xi) = \int_{-\infty}^\xi E[U](s)ds.$$

Note also that

$$\int_y^\xi [U(s - c\tau) - U(s)]ds = -c\int_y^\xi U'(s - \theta c\tau)d\theta \rightarrow -c\int_{0}^{1} U(\xi - \theta c\tau)d\theta,$$  

(3.6)

as $y \to -\infty$. Then, letting $y \to -\infty$ in (3.5), we obtain

$$cU(\xi) + \frac{\partial_2}{2} \int_{0}^{1} U(\xi - \theta c\tau)d\theta \geq \int_{-\infty}^{\xi} E[U](s) + \frac{\partial_1}{4} \int_{-\infty}^{\xi} U(s)ds,$$

which implies that $U(\xi)$ is integrable on $(-\infty, \xi']$, and hence $U(\xi - c\tau)$ is also integrable on $(-\infty, \xi']$. Moreover, we have

$$cU(\xi) = \int_{-\infty}^{\xi} E[U](s)ds + \int_{-\infty}^{\xi} f(U(s), U(s - c\tau))ds.$$  

(3.7)

Step 2. We show that $U(\xi) = O(e^{\gamma\xi})$ as $\xi \to -\infty$ for some $\gamma > 0$. This is achieved by showing the function $V(\xi) = \int_{-\infty}^{\xi} U(s)ds$ satisfying $V(\xi) = O(e^{\gamma\xi})$ as $\xi \to -\infty$ for some $\gamma > 0$.

It is easy to see that $V(\xi)$ is increasing and satisfies $V(-\infty) = 0$. Integrating (3.4) from $-\infty$ to $\xi$, $\xi \leq \xi'$, we get

$$cU(\xi) \geq \int_{-\infty}^{\xi} E[U](s)ds + \frac{\partial_1}{4} V(\xi) + \frac{\partial_2}{2} [V(\xi - c\tau) - V(\xi)].$$  

(3.8)
Similar to (3.6), we have \( \int_{y}^{\xi}[V(s - c\tau) - V(s)]ds \to -c\tau \int_{0}^{\xi}V(\xi - \theta c\tau)d\theta \). Noting that
\[
\lim_{y \to -\infty} \int_{y}^{\xi} E[U](s)ds = \lim_{y \to -\infty} \int_{y}^{\xi} E[V](s)ds = D \sum_{k=1}^{n} \left[ \int_{\xi}^{\xi + \alpha} V(s)ds - \int_{\xi - \alpha}^{\xi} V(s)ds \right] \geq 0,
\]
then for any \( \xi \leq \xi' \), it follows from (3.8) that
\[
cV(\xi) + \frac{\rho_2 c\tau}{2} \int_{0}^{\xi} V(\xi - \theta c\tau)d\theta \geq \frac{\rho_1}{4} \int_{-\infty}^{\xi} V(s)ds = \frac{\rho_1}{4} \int_{-\infty}^{0} V(s + \xi)ds.
\]
which implies that
\[
cV(\xi) + \left[ \frac{\rho_2 c\tau}{2} \right] \int_{0}^{\xi} V(\xi - \theta c\tau)d\theta \geq \frac{\rho_1}{4} \int_{-\infty}^{0} V(s + \xi)ds \geq \frac{\rho_1}{4} rV(\xi - r).
\]
In view of \( V \) increasing, for any \( r > 0 \) and \( \xi \leq \xi' \),
\[
\left( c + \frac{\rho_2 c\tau}{2} \right) V(\xi) \geq \frac{\rho_1}{4} \int_{-\infty}^{0} V(s + \xi)ds \geq \frac{\rho_1}{4} rV(\xi - r).
\]
Choose \( r_0 > 0 \) sufficiently large such that \( \theta_0 = (4c + 2|\rho_2 c\tau|/(\rho_1 r_0)) \in (0, 1) \). Thus, \( V(\xi - r_0) \leq \theta_0 V(\xi) \) for all \( \xi \leq \xi' \). Define \( W(\xi) = V(\xi)e^{-\gamma \xi} \), where \( \gamma = (1/r_0) \ln(1/\theta_0) \).
Then for any \( \xi \leq \xi' \), \( W(\xi - r_0) \leq W(\xi) \). Hence, \( 0 \leq W(\xi) \leq K := \max\{ W(s) | s \in [\xi' - r_0, \xi'] \} \) for \( \xi \leq \xi' \), it follows that \( V(\xi) = O(e^{\gamma \xi}) \) as \( \xi \to -\infty \).
Using (3.7) we have
\[
cU(\xi) = D \sum_{k=1}^{n} \left[ V(\xi + \sigma_k) + V(\xi - \alpha_k) - 2V(\xi) \right] + \int_{-\infty}^{\xi} f(U(s), U(s - c\tau))ds.
\]
Noting that
\[
0 \leq \int_{-\infty}^{\xi} f(U(s), U(s - c\tau))ds \leq \delta_1 f(0, 0)V(\xi) + \delta_2 f(0, 0)V(\xi - c\tau),
\]
we obtain \( U(\xi) = O(e^{\gamma \xi}) \) as \( \xi \to -\infty \).

**Step 3.** For \( 0 < \Re \lambda < \gamma \), we define a two-sided Laplace transform of \( U \) by
\[
L(\lambda) = \int_{-\infty}^{+\infty} U(\xi)e^{-\lambda \xi}d\xi.
\]
Rewrite (2.1) as
\[
cU'(\xi) - E[U](\xi) - \delta_1 f(0, 0)U(\xi) - \delta_2 f(0, 0)U(\xi - c\tau)
= f(U(\xi), U(\xi - c\tau)) - \delta_1 f(0, 0)U(\xi) - \delta_2 f(0, 0)U(\xi - c\tau).
\] (3.9)
Multiplying both sides of (3.9) by $e^{-\lambda \xi}$ and integrating along $\xi$ on $\mathbb{R}$, we get

$$\Delta_0(c, \lambda) L(\lambda) = \int_{-\infty}^{+\infty} [f(U(\xi), U(\xi - c \tau)) - \partial_1 f(0, 0) U(\xi) - \partial_2 f(0, 0) U(\xi - c \tau)] e^{-\lambda \xi} d\xi.$$  

Let $u(\xi) = U(-\xi)$ and $\lambda = -\lambda$ in the above equality, then $u(+\infty) = 0$ and

$$\Delta_0(c, -\lambda) L_1(\lambda) = \int_{-\infty}^{+\infty} h(\xi) e^{-\lambda \xi} d\xi,$$

(3.10)

where $L_1(\lambda) = \int_{-\infty}^{+\infty} u(\xi) e^{-\lambda \xi} d\xi$ and

$$h(\xi) = f(u(\xi), u(\xi + c \tau)) - \partial_1 f(0, 0) u(\xi) - \partial_2 f(0, 0) u(\xi + c \tau).$$

Since $u(+\infty) = 0$, $h(\xi) = O(u^2(\xi) + u^2(\xi + c \tau))$ as $\xi \to +\infty$, the right-hand of (3.10) is well defined for $-2 \gamma < \text{Re} \lambda < 0$. Now, we use a property of Laplace transform (Widder [20, p.89]). Since $u(\xi) > 0$, there exists a real number $\vartheta$ such that $L_1(\lambda)$ is analytic for $\vartheta < \text{Re} \lambda < 0$ and $L_1(\lambda)$ has a singularity at $\lambda = \vartheta$. Hence $L_1(\lambda)$ is well defined until $\lambda$ is a zero of $\Delta_0(c, -\lambda) = 0$. It follows from Lemma 2.1 that $L_1(\lambda)$ is well defined for $-\lambda_1(c) < \text{Re} \lambda < 0$.

From (3.10) we have

$$F(\lambda) := \int_{-\infty}^{+\infty} u(\xi) e^{-\lambda \xi} d\xi = \frac{\int_{-\infty}^{+\infty} h(\xi) e^{-\lambda \xi} d\xi}{\Delta_0(c, -\lambda)} - \int_{-\infty}^{0} u(\xi) e^{-\lambda \xi} d\xi.$$  

In order to apply Lemma 3.1, we define $J(\lambda) = F(\lambda)(\lambda + \lambda_0)^{(k+1)}$, where $k = 0$ if $c > c_*$ and $k = 1$ if $c = c_*$. It is easy to verify that $J$ is analytic in the strip $\{ \lambda \in \mathbb{C} | -\lambda_1(c) \leq \text{Re} \lambda < 0 \}$.

Using Lemma 3.1, the rest of the proof is similar to that of [21, Theorem 4.8] and is omitted. This completes the proof. 

\[ \blacksquare \]

**Corollary 3.3.** Let the assumptions of Theorem 3.2 be satisfied. Then, for all $c \geq c_*$,

$$\lim_{\xi \to -\infty} \frac{U'(\xi)}{U(\xi)} = \lambda_1(c) \quad \text{and} \quad \lim_{\xi \to +\infty} \frac{U'(\xi)}{U(\xi) - K} = \lambda_3(c).$$

**Proof of Theorem 2.4 (ii).** We now show that for $0 < c < c_*$, (1.1) has no travelling wave satisfying (1.2). Suppose on the contrary that (1.1) has a travelling wave $U$ connecting $0$ and $K$. From Lemma 2.3, $\Delta_0(c, -\lambda)$ has no real zeroes, it follows that $L_1(\lambda)$ is defined for $\lambda$ with $\text{Re} \lambda < 0$. Also, from (3.10),

$$\int_{-\infty}^{+\infty} [\Delta_0(c, -\lambda) u(\xi) - h(\xi)] e^{-\lambda \xi} d\xi = 0.$$  

By $\Delta_0(c, -\lambda) \to -\infty$ as $\lambda \to -\infty$, we obtain a contradiction, and the assertion follows. \[ \blacksquare \]

4. Monotonicity and uniqueness of travelling waves

We first establish a strong comparison principle.
THEOREM 4.1. For any fixed $\sigma \in \mathbb{R}^n$ with $|\sigma| = 1$, let $(U_1, c, \sigma)$ and $(U_2, c, \sigma)$ be two travelling waves of (1.1) with direction $\sigma$ and speed $c \geq c_\sigma(\sigma)$ satisfying $U_1(\cdot) \geq U_2(\cdot)$ on $\mathbb{R}$. Then either $U_1 \equiv U_2$ or $U_1 > U_2$ on $\mathbb{R}$.

Proof. Let $L_1, \mu$ and $H$ be defined as in the proof of Lemma 2.5 (ii). Thus

$$U_i(\xi) = \frac{1}{c} e^{-\frac{\mu}{s}} \int_{-\infty}^{\xi} e^{\frac{s}{c}} H[U_i](s) \, ds, \quad i = 1, 2.$$ 

Suppose that there exists $\xi_0 \in \mathbb{R}$ such that $U_1(\xi_0) = U_2(\xi_0)$. Then,

$$0 = cU_1(\xi_0) - cU_2(\xi_0) = e^{\frac{-\mu}{c}} \int_{-\infty}^{\xi_0} \frac{\mu}{c} s \{ H[U_1](s) - H[U_2](s) \} \, ds.$$ 

As $H[U_1](\cdot) \geq H[U_2](\cdot)$, $H[U_1](s) = H[U_2](s)$ for $s \leq \xi_0$. Thus $U_1(s) = U_2(s)$ for $s \leq \xi_0$. Therefore, for $s \leq \xi_0$,

$$0 = H[U_1](s) - H[U_2](s) \geq D \sum_{k=1}^{n} [U_1(s + \sigma_k) - U_2(s - \sigma_k)] \geq 0,$$

it follows that $U_1(s) = U_2(s)$ for $s \leq \xi_0 + \sigma_k$, where $\sigma_{k_0} = \max_{k=1,\ldots,n} \{ \sigma_k \} > 0$. Repeating the argument, we obtain $U_1 \equiv U_2$ on $\mathbb{R}$. This completes the proof.

THEOREM 4.2 [MONOTONICITY]. For any fixed $\sigma \in \mathbb{R}^n$ with $|\sigma| = 1$, let $(U, c, \sigma)$ be a travelling wave of (1.1) with direction $\sigma$ and speed $c \geq c_\sigma(\sigma)$. Then $U(\cdot) > 0$.

Proof. From Corollary 3.3, there exist $x_1, x_2 > 0$ such that $U'(\xi) > 0$ for $\xi \geq x_1$ and $\xi \leq -x_2$. Define $\eta^* := \inf \{ \eta > 0 | U(\xi + s) \geq U(\xi), \forall \xi \in \mathbb{R}, s \geq \eta \}$. Since $U(\cdot) < K$ and $U(+\infty) = K$, $\eta^*$ is well defined. Also, $U(\xi + s) \geq U(\xi)$ for $\xi \in \mathbb{R}$ and $s \geq \eta^*$.

By virtue of Theorem 4.1 and using a method similar to that of [5, Lemma 4.3], one can show that $\eta^* = 0$, which implies that $U'(\xi) \geq 0$ for all $\xi \in \mathbb{R}$. It then follows from Lemma 2.5 (ii) that $U'(\xi) > 0$ for all $\xi \in \mathbb{R}$. This completes the proof.

To prove the uniqueness of travelling waves, we need two important lemmas.

LEMMA 4.3. For any fixed $\sigma \in \mathbb{R}^n$ with $|\sigma| = 1$, let $(U, c, \sigma)$ be a travelling wave of (1.1) with direction $\sigma$ and speed $c \geq c_\sigma(\sigma)$. Then there exists $\rho_0 = \rho_0(c, f) \in (0, 1)$ such that for any $\rho \in (0, \rho_0]$ and $\xi \in \{ \xi | U(\xi) > K - \rho_0 \}$,

$$B(\rho, \xi) := (1 + \rho)f(U(\xi), U(\xi - c\tau)) - f((1 + \rho)U(\xi), (1 + \rho)U(\xi - c\tau)) > 0.$$ 

Proof. Note that $U(+\infty) = K$, $B(0, \xi) = 0$ and

$$B(\rho, \xi)|_{\rho = 0} = f(U(\xi), U(\xi - c\tau)) - \partial_1 f(U(\xi), U(\xi - c\tau)) U(\xi) - \partial_2 f(U(\xi), U(\xi - c\tau)) U(\xi - c\tau)$$

$$\rightarrow -[\partial_1 f(K, K) + \partial_2 f(K, K)] K > 0,$$

as $\xi \rightarrow +\infty$. Based on these observations, it is easy to see that the assertion holds.

\[\square\]
For a given travelling wave $U$ of (1.1), define

$$\kappa = \kappa(U) := \sup \left\{ \frac{U(\xi)}{U'(\xi)} \mid U(\xi) \leq K - \rho_0 \right\}.$$ 

Clearly, $0 < \kappa < +\infty$, since $\lim_{\xi \to -\infty} U''(\xi)/U'(\xi) = \lambda_1(c) > 0$ and $U''(\cdot) > 0$ on $\mathbb{R}$.

**Lemma 4.4.** For any fixed $\sigma \in \mathbb{R}^n$ with $|\sigma| = 1$, let $(U, c, \sigma)$ and $(V, c, \sigma)$ be two travelling waves of (1.1) with direction $\sigma$ and speed $c \geq c_*(\sigma)$, and $\kappa = \kappa(U)$. If there exists $\rho \in (0, \rho_0]$ such that $(1 + \rho)U(\cdot - \kappa \rho) \equiv V(\cdot)$ on $\mathbb{R}$, then $U(\cdot) \equiv V(\cdot)$ on $\mathbb{R}$.

**Proof.** Define $W(\rho, \xi) = (1 + \rho)U(\xi - \kappa \rho) - V(\xi)$ and

$$\rho^* = \inf \{ \rho > 0 \mid W(\rho, \xi) \equiv 0, \forall \xi \in \mathbb{R} \}.$$ 

By continuity, $W(\rho^*, \xi) \equiv 0$, for all $\xi \in \mathbb{R}$. Thus, it is sufficient to show that $\rho^* = 0$.

Suppose for contradiction that $\rho^* \in (0, \rho_0]$. By the definition of $\kappa$ and Theorem 4.2, we have

$$\frac{\partial}{\partial \rho} W(\rho, \xi) = U(\xi - \kappa \rho) - \kappa (1 + \rho)U'(\xi - \kappa \rho) < 0$$

on $\{ \xi \mid U(\xi - \kappa \rho) \leq K - \rho_0 \}$. Also noting that $W(\rho^*, \infty) = \rho^* K > 0$. Hence, there exists $\xi_0 \in \mathbb{R}$ with $U(\xi_0 - \kappa \rho^*) > K - \rho_0$ such that the function $W(\rho^*, \xi)$ attains its minimal at the point $\xi_0$, i.e. $W(\rho^*, \xi_0) = W(\rho^*, \xi_0) = 0$. Then $U(\xi) = V(\xi)$,

$$(1 + \rho^*)U'(P_0) = V'(\xi_0), \quad (1 + \rho^*)U(P_0 + \sigma_k) \equiv V(\xi_0 + \sigma_k), k = 1, \cdots, n$$

and $(1 + \rho^*)U(P_0 - c\tau) \equiv V(\xi_0 - c\tau)$, where $P_0 = \xi_0 - \kappa \rho^*$. So, we have

$$0 = cV'(\xi_0) - E[V](\xi_0) - f(V(\xi_0), V(\xi_0) - c\tau)$$

$$\geq c(1 + \rho^*)U'(P_0) - E[(1 + \rho^*)U(P_0)] - f((1 + \rho^*)U(P_0), (1 + \rho^*)U(P_0 - c\tau))$$

$$= (1 + \rho^*)f(U(P_0), U(P_0 - c\tau)) - f((1 + \rho^*)U(P_0), (1 + \rho^*)U(P_0 - c\tau))$$

$$> 0, \quad \text{by Lemma 4.3}$$

which is a contradiction. Hence, $\rho^* = 0$, and the assertion of this lemma follows. \qed

**Theorem 4.5 [Uniqueness].** For any fixed $\sigma \in \mathbb{R}^n$ with $|\sigma| = 1$, let $(U_1, c, \sigma)$ and $(U_2, c, \sigma)$ be two travelling waves of (1.1) with direction $\sigma$ and speed $c \geq c_*(\sigma)$. Then there exists $\xi_0 \in \mathbb{R}$ such that $U_1(\cdot + \xi_0) \equiv U_2(\cdot)$.

**Proof.** Based on Theorem 4.1 and Lemma 4.4, the proof is similar to those of [5, Theorem 5.1] and [13, Theorem 2] and is omitted. \qed

Using the uniqueness result, we obtain the periodicity and symmetry of travelling waves. We only consider the case for $n = 2$. Take $\sigma = (\cos \theta, \sin \theta)$, $\theta \in \mathbb{R}$, and denote $U_{c, \theta}(\cdot) := (U(\cdot), c, \theta)$. 
Theorem 4.6 [Periodicity and symmetry]. For \( c \geq c_*(\theta), U_c(\theta), \theta \in \mathbb{R} \), is a periodic function of period \( \pi/2 \) with respect to \( \theta \) and has the symmetry in \( \theta = \pi/4 \) with phase shift, that is for any \( \theta \in \mathbb{R} \), there exist \( \xi_0 = \xi_0(\theta), \xi_1 = \xi_1(\theta) \in \mathbb{R} \) such that
\[
U_{c,\theta+(\pi/2)}(\cdot + \xi_0) = U_{c,\theta}(\cdot) \quad \text{and} \quad U_{c,(\pi/2)-\theta}(\cdot + \xi_1) = U_{c,\theta}(\cdot).
\]

Proof. We only prove that for any \( \theta \in \mathbb{R} \), there exist \( \xi_1 = \xi_1(\theta) \in \mathbb{R} \) such that \( U_{c,(\pi/2)-\theta}(\cdot + \xi_1) = U_{c,\theta}(\cdot) \). From Theorem 2.6, \( c_* := c_*(\theta) = c_*(\pi/2 - \theta) \). Thus, it is sufficient to show that for \( c \geq c_* \), \( U_{c,(\pi/2)-\theta}(\cdot + \xi_1) = U_{c,\theta}(\cdot) \). For convenience, we denote \( U_1(\cdot) = U_{c,(\pi/2)-\theta}(\cdot) \) and \( U_2(\cdot) = U_{c,\theta}(\cdot) \).

For any \( \theta \in \mathbb{R} \), it follows from (2.4) and (2.5) that
\[
cU_1'(\xi) = \frac{1}{2} \left[ U_1(\xi + \cos \theta) + U_1(\xi - \cos \theta) + U_1(\xi + \sin \theta) \right]
+ U_1(\xi - \sin \theta) - 4U_1(\xi) + f(U_1(\xi), U_1(\xi - c\theta))
\]
and
\[
cU_2'(\xi) = \frac{1}{2} \left[ U_2(\xi + \cos \theta) + U_2(\xi - \cos \theta) + U_2(\xi + \sin \theta) \right]
+ U_2(\xi - \sin \theta) - 4U_2(\xi) + f(U_2(\xi), U_2(\xi - c\theta)).
\]

Obviously, (4.2) is equivalent to
\[
cU_2'(\xi) = \frac{1}{2} \left[ U_2(\xi + \cos \theta) + U_2(\xi - \cos \theta) + U_2(\xi + \sin \theta) + U_2(\xi - \sin \theta) - 4U_2(\xi) \right]
+ f(U_2(\xi), U_2(\xi - c\theta)).
\]

By the uniqueness of the travelling fronts, \( U_1(\cdot + \xi_1) = U_2(\cdot) \) for some \( \xi_1 = \xi_1(\theta) \in \mathbb{R} \). Then \( U_{c,(\pi/2)-\theta}(\cdot + \xi_1) = U_{c,\theta}(\cdot) \) for any \( \theta \in \mathbb{R} \). This completes the proof. \( \square \)

5. Applications

In the previous sections, we studied the properties of the travelling fronts of the \( n \)-dimensional lattice delayed differential equation (1.1). Clearly, our results are easily extended to the following equation with multi-delays
\[
u_\eta(t) = D(\Delta_\eta u_\eta) + f(u_\eta(t), S_1(u_\eta(t - \tau_1)), \ldots, S_m(u_\eta(t - \tau_m))), \quad \eta \in \mathbb{Z}^n, \quad t > 0,
\]
where \( \tau_j \geq 0 \) and \( S_j(\cdot) (j = 1, \ldots, m) \) are given constants and functions, respectively.

Example 5.1. Consider the equation
\[
u_\eta(t) = D(\Delta_\eta u_\eta) + u_\eta(t - \tau)[1 - u_\eta(t)], \quad \eta \in \mathbb{Z}^n, \quad t > 0,
\]
which was derived from branching theory in [15]. As an application of the results in [26], Zou obtained the existence of travelling fronts of (5.1) by constructing a pair of sub- and supersolutions. More precisely, they showed that for any fixed \( \sigma \in \mathbb{R}^n \) with \( |\sigma| = 1 \), there exists a number \( c_*(\sigma) > 0 \) such that for each \( c > c_*(\sigma) \), (5.1) has a travelling front \((U(\cdot), c, \theta)\) connecting 0 and \( K := 1 \).
Let $f(u, v) = u[1 - v]$, then (A$_1$) and (A$_2$) hold. By Theorem 2.4 we see that the condition $c \geq c_*(\sigma)$ is not only a sufficient condition but also a necessary condition for the existence of travelling fronts. Also, by Theorems 4.2 and 4.5, any travelling wave with given direction $\sigma$ and speed $c \geq c_*(\sigma)$ is strictly monotone increasing and unique (up to a translation). Obviously, our results extend and complement those established by Zou [26].

**Example 5.2.** Consider the equation

$$u_i'(t) = D[u_{i+1}(t) + u_{i-1}(t) - 2u_i(t)] - du_i(t) + b(u_i(t - \tau)), \quad i \in \mathbb{Z}, \quad t > 0,$$  \hspace{1cm} (5.2)

where $d > 0$ is a constant and $b$ is a function satisfying

- (B$_1$) $b \in C^2([0, K], \mathbb{R})$, $b(0) = b(K) - dK = 0$, $b'(K) < d$ and $b(u) > du$ for any $u \in (0, K)$, where $K > 0$ is a constant;
- (B$_2$) $b'(u) \geq 0$ and $b(u) \leq b'(0)u$ for all $u \in [0, K]$.

Let $\Delta(c, \lambda) = c\lambda - D[e^\lambda + e^{-\lambda} - 2] + d - b'(0)e^{-\lambda c\tau}$, then there exist $\lambda_*, > 0$ and $c_* > 0$ such that $\Delta(c_*, \lambda_*) = (\partial / \partial \lambda) \Delta(c_*, \lambda)|_{\lambda=\lambda_*} = 0$.

Under the monostable assumptions (B$_1$) and (B$_2$), Ma and Zou [16] showed that (i) for every $c \geq c_*$, (5.2) has a travelling wavefront $(U(\xi), c)$, and (ii) for any $c \in (0, c_*)$, (5.2) has no travelling waves. Furthermore, they showed that if $(\bar{U}(\xi), c > c_*)$ is another travelling wave solution of (5.2) with

$$\lim_{\xi \to -\infty} \bar{U}(\xi)e^{-\lambda_1(c)\xi} < +\infty,$$  \hspace{1cm} (5.3)

where $\lambda_1(c)$ is the smallest root of the equation $\Delta(c, \lambda) = 0$, then there exists $\xi_0 \in \mathbb{R}$ such that $\bar{U}(\xi) = U(\xi + \xi_0)$. It is natural to ask whether the uniqueness holds true without condition (5.3) and whether the uniqueness holds true for the case $c = c_*$?

Let $n = 1$ and $f(u, v) = -du + b(v)$, one can easily verify that (A$_1$) and (A$_2$) hold. From Theorem 3.2, (5.3) holds. In fact, we have the following result.

**Theorem 5.1.** Let $(U(\cdot), c)$ be a travelling wave solution of (5.2) with speed $c \geq c_*$. Then for $c > c_*$,

$$\lim_{\xi \to -\infty} U(\xi)e^{-\lambda_1(c)\xi} = a_0(c),$$  \hspace{1cm} (5.4)

and for $c = c_*$,

$$\lim_{\xi \to -\infty} U(\xi)\xi^{-1}e^{-\lambda_1(c_*)\xi} = a_0(c_*),$$  \hspace{1cm} (5.5)

where $a_0(c)$ and $a_0(c_*)$ are positive constants.

Moreover, by Theorems 4.2 and 4.5, the following results hold.

**Theorem 5.2.**

(i) Let $(U(\cdot), c)$ be a travelling wave of (5.2) with speed $c \geq c_*$. Then $U'(\cdot) > 0$.

(ii) Let $(U_1(\cdot), c)$ and $(U_2(\cdot), c)$ be two travelling wave solutions of (5.2) with speed $c \geq c_*$. Then there exists $\xi_0 \in \mathbb{R}$ such that $U_1(\cdot + \xi_0) = U_2(\cdot)$.  


Remark 5.3. Theorem 5.1 shows that all travelling waves of (5.2) with speed $c > c_*$ and $c = c_*$ possess the prior asymptotic behaviour (5.4) and (5.5), respectively. Also, the result on the uniqueness of the travelling waves in Theorem 5.2 is valid not only for $c > c_*$ but also for $c > c_*$. Obviously, Theorems 5.1 and 5.2 improve and complement the related results in [16].

We remark that the stability of travelling fronts of (5.2) is also obtained by Ma and Zou [16] through a related continuum equation by extending the spatial variable from $i \in \mathbb{Z}$ to $x \in \mathbb{R}$. However, the stability of travelling fronts of the $n - D$ lattice equation (1.1) seems to be an interesting and challenging problem, and we leave this for future research.

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