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On the conditional default probability in a regulated market: a structural approach

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In this article, we consider a regulated market and explore the default events. By using a so-called reflected Ornstein–Uhlenbeck process with two-sided barriers to formulate the price dynamics, we derive the expression for the conditional default probability. In the cases of a single observation and multiple observations, the conditional default probabilities are explicitly expressed in terms of the inverse Laplace transforms. Finally, we present a numerical simulation associated with the conditional default probability.

Keywords: Conditional default probability; Reflected Ornstein–Uhlenbeck process; Default risk; Inverse Laplace transform

1. Introduction

Conditional default probability (CDP) is one of the most important concepts in credit risk. In this paper, we focus on the CDP in a regulated (or controlled) market. As is known, in a regulated market, the goods or services (for instance, grain, water, gas, electricity supply and other important materials or services for a country) are usually regulated by a government-appointed body and the prices associated with these goods or services are allowed to be charged (see, e.g., WIKIPEDIA). The price control (both the price ceiling and the price floor are charged) commonly results in the boundedness of the prices of these regulated goods or services. This characteristic (boundedness) stimulated us to present a tractable bounded stochastic process to describe the price dynamics of the regulated goods or services. More precisely, we propose a so-called reflected Ornstein–Uhlenbeck (O–U) process on the regulated band $[0, b]$ (the points $0$ and $b > 0$ are two reflecting barriers) to model the price dynamics of the regulated goods or services. The reflection at point zero ensures that the price process remains non-negative, which is a natural setting in the real world (see, e.g., Goldstein and Keirstead 1997). The reflection at point $b$ is to ensure that the price is always no higher than the price ceiling $b$.

We consider the CDP in a regulated market, but we also have to face another determinant of the CDP; that is, market information. In most of the existing literature, market information is usually assumed to be ‘complete’. However, in reality, we actually only receive ‘incomplete’ (or ‘partial’) information from real-world observations. Typically, for instance, incomplete information concerning the asset can be interpreted as the quarterly provided reports on the asset evaluation of the firm (see, e.g., Duffie and Lando 2000). In the present setting, we assume that we can only observe the price process at some discrete time points (as partial information).

Currently, there are two popular modeling approaches to deal with default (or credit risk): the structural approach and the reduced-form approach. The former is due to Merton (1974), who first introduced the first passage argument for considering the default of a single entity. The asset dynamics of the firm is described mathematically by a geometric Brownian motion or a diffusion process, and default occurs when the value of the firm down-crosses a certain threshold level. Zhou (2000) further suggested that the asset process follows a jump diffusion, and the default occurs due to the jumps of the underlying asset. This seems to be intuitive. In the reduced-form approach, the default is usually
characterized as the first jump of a Poisson process with a random intensity. A typical model for the default intensity is the so-called ‘affine process’ model (Duffie et al. 2003). Here, we employ the structural approach. That is, the default is due to down-crossing a certain threshold level by the price process of the regulated goods or services.

We study CDPs under a single observation and multiple observations, and we shall give their explicit formulas in terms of inverse Laplace transforms. Further, in the final part of the article, we present a numerical illustration. Our test involves the computation of the inverse Laplace transforms. A so-called ‘GWR’ algorithm (see, e.g., Valko and Abate 2004) is applied to numerically calculate the inverse Laplace transforms. The algorithm allows us to evaluate the inverse Laplace transforms by calculating the series of the Gaver-functional. Furthermore, the convergence of the series can be accelerated by virtue of the WYNN-RHO convergence acceleration scheme. The test results will be outlined in section 4.

The rest of the paper is organized as follows. In section 2, a model description is given. The conditional survival probability under a single observation and multiple observations is established in section 3. Section 4 is devoted to presenting a numerical illustration, and concluding remarks are given in section 5. All proofs of theorems and lemmas are given in the appendix.

2. Model description

In this section we start with a mathematical description of the model. Let \( \Lambda = (\Omega, \mathcal{G}, (\mathcal{G}_t)_{t \geq 0}, \mathbf{P}) \) be a complete probability space with \((\mathcal{G}_t)_{t \geq 0}\) satisfying the usual conditions. \( \mathbf{P} \) is the physical (statistical) measure. Suppose that the market price (of some asset, for example one of the regulated goods of services) follows a bounded process \( Q = (Q_t)_{t \geq 0} \) a one-dimensional reflected O-U process with two-sided barriers:

\[
\begin{align*}
&\{dQ_t = (\mu - \sigma Q_t)dt + \sigma dw_t, t \geq 0,\}
&\{Q_0 = v \in [0,b],\}
\end{align*}
\]

where \( w = (w_t)_{t \geq 0} \) is a standard Brownian motion and \( \mu, \sigma \in \mathbb{R}^+ \). \( l = (l_t)_{t \geq 0} \) and \( u = (u_t)_{t \geq 0} \) are usually called the regulators of the reflected process \( Q \) respectively at points 0 and \( b \). Moreover, \( l \) and \( u \) are uniquely determined by the following (see, e.g., Harrison 1986 and Bo et al. 2006).

- For \( t \in [0, \infty) \), the sample paths \( t \rightarrow l_t \) and \( t \rightarrow u_t \) are continuous non-decreasing and \( l_0 = u_0 = 0 \).

\[\int_0^t I_{[Q_t \geq 0]}dl_s = 0, \quad \text{and} \quad \int_0^t I_{[Q_t < b]}du_s = 0, \quad \text{for all} \ t > 0.\]

We then say that the process \( Q = (Q_t)_{t \geq 0} \) is a reflected process with reflecting barriers 0 and \( b \). Obviously, \( Q_t \in [0, b] \) for all \( t \geq 0 \), provided \( Q_0 \in [0, b] \).

For the structural approach, we define the default time \( \tau \) (as a random time, not essentially a stopping time) by

\[\tau = \inf\{t \geq 0; \ Q_t \leq d\}, \quad (2.2)\]

where \( d \in (0, b) \) denotes the default barrier. Let \( D_t := I_{[\tau \leq t]} \), and we call \((D_t)_{t \geq 0}\) the default indicator process.

Assume that \( 0 \leq t_1 < t_2 < \cdots < t_n < \cdots \) are a sequence of deterministic observed times that can be interpreted as the moments when the quarterly reports are provided by the firm. For each \( t > 0 \) fixed, define \( n_t := \max\{j; t_j \leq t\} \). We denote the observed price at time \( t \) by \( Y_t := Q_t + \xi_t \), where \( \xi = (\xi_t)_{t \geq 0} \) is an extra noisy source\( \xi \) independent of the price process \( Q \). The partial information induced as in Duffie and Lando (2000) is \( \mathcal{F} = (\mathcal{F}_t)_{t \geq 0} \subset \mathcal{G} \), where

\[\mathcal{F}_t = \sigma(Y_{t_1}, \ldots, Y_{t_n}) \lor \sigma(D_{t_0}; 0 \leq u \leq t). \quad (2.3)\]

On the other hand, following Bielecki and Rutkowski (2002), it follows that

\[\mathcal{F}_t = \sigma((Y_{t_1}, \ldots, Y_{t_n})) \lor \sigma([\tau \leq u; \ u \leq t]), \quad (2.4)\]

and, for every \( t > 0 \),

\[\mathcal{F}_t \cap \{\tau > t\} = \sigma((Y_{t_1}, \ldots, Y_{t_n})) \cap \{\tau > t\}. \quad (2.5)\]

Obviously, from (2.4), the default time \( \tau \) defined in (2.2) is an \( \mathcal{F} \)-stopping time. For every \( (t, s) \in [0, \infty) \times [0, \infty) \) with \( t \geq s \), define the Conditional Survival Probability by

\[\tilde{\ell}(s, t) := 1 - \mathbf{E}(D_t \mid \mathcal{F}_s) = \mathbf{P}(\tau > t \mid \mathcal{F}_s). \quad (2.6)\]

We conclude this section with a lemma that will be used in the following sections.

**Lemma 2.1:** Let \((\Omega, \mathcal{G}, \mathbf{P})\) be a probability space, \( A \in \mathcal{G} \) and \( \eta \) be an integrable random variable. Suppose that \( \mathcal{H} \) is a sub-\(\sigma\)-field of \( \mathcal{G} \) such that \( \mathcal{H} \cap A = \emptyset \cap A \), then

\[
\mathbf{E}(\eta 1_A \mid \mathcal{H}) = \frac{\mathbf{E}(\eta 1_A \mid \mathcal{H})}{\mathbf{E}(1_A \mid \mathcal{H})}. 
\]

3. The conditional survival probability (CSP)

In this section, we calculate the CSP \( \tilde{\ell}(s, t) \) defined in (2.6) for the case \( t > s \) and \( s = t_i \) with \( i = 1, 2, \ldots, n \) (\( n \in \mathbb{N} \)).
We separately consider the cases of a single observation and multiple observations in the following two subsections.

### 3.1. The single observation

First, we consider the case \( s=t_1 \) (i.e. the case of a single observation). From Lemma 2.1, we obtain the following useful lemmas.

**Lemma 3.1:** Let \( n \in \mathbb{N} \). Suppose \( n_1=n \) and \( t>s \), then on the event \( \{\tau>s\} \),

\[
\tilde{\ell}(s, t) = \frac{P(\tau > t \mid Y_1, \ldots, Y_n)}{P(\tau > s \mid Y_1, \ldots, Y_n)}
\]

Recall the price process \( Q \) in (2.1). Define its transition distribution by

\[
p(t, x, d) := \frac{P(Q_t \in d \mid Q_0 = x)}{P(Q_t \in d)},
\]

(3.1)

for time \( t>0 \). If \( X=(X_t)_{t \geq 0} \) is a \((G_t)_{t \geq 0}\)-adapted random process on \( \Lambda \), we denote its transition distribution by

\[
F_X(t; d) := P(X_t \in d), \quad t \geq 0.
\]

On the other hand, from Linetsky (2005), we have the following lemma.

**Lemma 3.2:** For \( t>0 \) and \( x, y \in [0, b] \), the transition density \( p(t, x, v) \) of the price process \( Q \) admits the following eigenfunction expansion:

\[
\hat{p}(t, x, v) := \frac{p(t, x, dv)}{dv} = \pi(v) + m(v) \sum_{n=1}^\infty \exp(-\lambda_n t) \phi_n(x) \varphi_n(v),
\]

(3.2)

where

\[
\pi(v) = \frac{\sqrt{2} \alpha}{\sigma} \Phi(z_v) - \Phi(x_v),
\]

\[
m(v) = \frac{2}{\sigma^2} \exp \left( -a(\mu/\alpha - v^2) \right)
\]

are, respectively, the stationary density and the speed density of \( Q \). \( \phi(\cdot) \) and \( \Phi(\cdot) \) are the respective standard normal density and cumulative distribution function. Here the eigenvalues \( \lambda_n \) and the eigenfunctions \( \varphi_n(\cdot) \) admit the following large-\( n \) approximations:

\[
\lambda_n = \frac{\alpha^2 \pi^2 n^2}{2b^2} + c_0 + O \left( \frac{1}{n^2} \right),
\]

\[
\varphi_n(v) = \pm \frac{\sigma}{\sqrt{b}} \exp \left( \frac{z_v}{4} \right) \left[ \cos \left( \frac{n \pi v}{b} \right) + \frac{b}{n \pi \sigma^2} f(v) \sin \left( \frac{n \pi v}{b} \right) \right] + O \left( \frac{1}{n^2} \right).
\]

with

\[
x_v := -\sqrt{2 \alpha \mu / \sigma}, \quad y_v := \sqrt{2 \alpha / \sigma} \left( b - \frac{\mu}{\alpha} \right), \quad z_v := \sqrt{2 \alpha / \sigma} \left( v - \frac{\mu}{\alpha} \right),
\]

\[
c_0 := \frac{\sigma}{2} + \frac{\alpha^2}{6 \sigma^2} \left[ 3 b^2 \frac{\mu}{\alpha} + 3 \left( \frac{\mu^2}{\alpha^2} \right) \right],
\]

\[
f(v) := \frac{\alpha^2}{2 \sigma^2} v^2 - \frac{\alpha \mu}{2 \sigma^2} v^2 + \left( \frac{\mu}{\sigma} \right)^2 \left( \frac{\mu}{\sigma} \right)^2.
\]

This lemma provides an efficient method for evaluating the transition density of the reflected process \( Q \) (see Linetsky (2005) and section 4).

Let \( g \in C^2([0, b]) \) and \( \lambda > 0 \). Define

\[
A^{(l)}(x) := \frac{\alpha^2}{2} g''(x) + (\mu - ax) g'(x) - \lambda g(x), \quad \text{for } x \in [0, b].
\]

Then, for the default time \( \tau \) defined in (2.2), we have the following lemma.

**Lemma 3.3:** Assume that \( v \in [0, b] \) and \( d \in [0, v] \). If, for each \( \lambda > 0 \) fixed, \( f^{(l)}(\cdot) \) is a (non-trivial) solution to the equation

\[
A^{(l)}(v) = 0, \quad v \in [0, b], \quad f^{(l)}(b) = 0,
\]

and if \( f^{(l)}(y) \neq 0 \) for \( 0 \leq y \leq v \), then for the probability distribution \( \eta(\cdot,d) := P(\tau \in dt)^\dagger \), its Laplace transform is given by

\[
\mathcal{L}(\eta)(\lambda) := E_{\eta}(e^{-\lambda \tau}) = \frac{\tilde{l}^{(l)}(y)}{\tilde{l}^{(l)}(d)},
\]

where

\[
\tilde{l}^{(l)}(y) = 2^{-\delta/2} e^{2 \delta} \exp \left( \frac{y^2}{4} \right) \gamma(\delta; z_x, \beta_x),
\]

(3.5)

where

\[
z_x = (2a)^{1/2} \left( \frac{x - \mu}{\alpha} \right), \quad \beta_x = \frac{2a}{\sigma} \left( x - \frac{\mu}{\alpha} \right), \quad \delta = -\frac{\lambda}{\alpha},
\]

and \( \gamma(\delta; u, v) \) is given by

\[
\gamma(\delta; u, v) = \delta E^{(0)}_d (u) E^{(1)}_{\delta-1}(v) + 2 E^{(0)}_d (u) E^{(0)}_{\delta-1}(v),
\]

with

\[
E^{(0)}_d (u) := \sqrt{2} e^{-x^2/4} F_1 \left( -\frac{1}{2}; \frac{1}{2}; \frac{1}{2} \mu^2 \right),
\]

\[
E^{(1)}_d (u) := 2 x e^{-x^2/4} F_1 \left( \frac{1}{2} (1 - \delta); \frac{3}{2}; \frac{1}{2} \mu^2 \right).
\]

†An explicit expression for the inverse Laplace transform \( \eta_\beta \) of \( \mathcal{L}(\eta)_t \) does not seem to be possible to obtain. Instead, there are algorithms available for the numerical inversion of Laplace transforms (see Valkó and Vajda 2002, Valkó and Abate 2004 and section 4).
Here, \( _1F_1(a; x; x) \) is the Kummer confluent hypergeometric function, defined as follows: for \( a \in \mathbb{C}, \ a \in \mathbb{C} \) and \( b \in \mathbb{C} \),
\[
_1F_1(a; x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{x^n}{n!},
\]
where \((a)_n := 1 \), \((a)_n := a(a + 1) \ldots (a + n - 1) \) are the Pochhammer symbols. The following lemma gives a solution to (3.4).

**Lemma 3.4:** For any \( \lambda > 0 \) and \( x \in [0, b) \), let the function \( \bar{f}^{(a)} \) on \([0, b)\) be defined by (3.5). Then \( \bar{f}^{(a)} \) is a solution of equation (3.4).

We will use \( \mathcal{L}^{-1} \) to denote the inverse Laplace transform. The following is the main result of this subsection.

**Theorem 3.5:** Let \( s = t_1 \) and \( t > s \). Recall the price process \( Q \) defined in (2.1). Assume that it has initial distribution \( \mu(\nu) \). Then, the Conditional Survival Probability (CSP) is
\[
\tilde{\ell}(s, t) = \int_0^t \left[ \int_y ^{\infty} \frac{\left( \bar{f}^{(a)} \right)(y)}{\bar{f}^{(b)}(d) } \right] (dr) h(dv, Y_s, s) f_s^y(hdv, Y_s, s),
\]
where the function \( \bar{f}^{(a)} \) is defined in (3.5) and
\[
\frac{h(dv, y, s)}{d^v} = \frac{F_y (s; y - v)}{F_Y (s; y )} \int_0^s \frac{p(s, u, v) \mu(du)}{f_s^u(hdv, Y_s, s)}.
\]

### 3.2. Multiple observations

In this subsection we extend our consideration of a single observation to the case of multiple observations at moments \( t_1, t_2, \ldots, t_n \), \( n \in \mathbb{N} \). We use the following notation. For \( n \in \mathbb{N} \), let

- \( y^{(a)} = (y_{t_1}, y_{t_2}, \ldots, y_{t_n}) \) denote an observation of \( V^{(a)} = (Q_{t_1}, Q_{t_2}, \ldots, Q_{t_n}) \).
- \( y^{(n)} = (y_{t_1}, y_{t_2}, \ldots, y_{t_n}) \) denote an observation of \( Y^{(n)} = (Y_{t_1}, Y_{t_2}, \ldots, Y_{t_n}) \), and

\[
d^{(a)} = (a_{t_1}, a_{t_2}, \ldots, a_{t_n}) \text{ and } d^{(n)} = (a_{t_1}, a_{t_2}, \ldots, a_{t_n}) \text{ denote an observation of } \xi^{(a)} = (\xi_{t_1}, \xi_{t_2}, \ldots, \xi_{t_n}) = y^{(a)} - v^{(a)}.
\]

Define
\[
\begin{align*}
&h_n(dv, t_n | y^{(n)}) := P (V_n^{(n)} \in dv, \tau > t_n | Y^{(n)} = y^{(n)}), \\
g_n(dv, t_n | y^{(n)}) := P (V_n^{(n)} \in dv | Y^{(n)} = y^{(n)}, \tau > t_n), \\
&\hat{p}_n(v, t_n | y_{t_n}) := \frac{P (Q_{t_n} = y_{t_n} | Q_{t_{n-1}} = y_{t_{n-1}})}{dv}, \ 2 \leq i \leq n, \\
&\hat{p}_n (a_{t_i} | a_{t_{i-1}}) := \frac{P (\xi_{t_i} = a_{t_i} | \xi_{t_{i-1}} = a_{t_{i-1}})}{da_i}, \ 2 \leq i \leq n, \\
&\hat{p}_n (y_{t_i} | y_{t_{i-1}}) := \frac{P (Y_{t_i} = y_{t_i} | Y_{t_{i-1}} = y_{t_{i-1}})}{dy_i}, \ 2 \leq i \leq n.
\end{align*}
\]

Similarly as in the case of a single observation, in order to derive \( \hat{\ell}(s, t) \) for \( t > s \) and \( s = t_n \), it suffices to give \( h_n (dv, t_n | y^{(n)}) \) and \( g_n (dv, t_n | y^{(n)}) \) defined in (3.7). We obtain the following theorem.

**Theorem 3.6:** For \( n \in \mathbb{N} \) fixed, let \( h_n (dv, t_n | y^{(n)}) \) and \( g_n (dv, t_n | y^{(n)}) \) be defined by (3.7). Then we have
\[
\begin{align*}
&h_n (dv, t_n | y^{(n)}) \\
&= h_n (dv, t_n | y^{(n-1)}) \\
&\quad - \int_0^{t_n} \int_0^{t_n} \left[ \int_y ^{\infty} \frac{\left( \bar{f}^{(a)} \right)(y)}{\bar{f}^{(b)}(d) } \right] (dr) \mu(du) \\
&\quad \times F_y (s; y - v), \quad i = 2, \ldots, n, \quad (3.8)
\end{align*}
\]

where \( \Delta t_i := t_i - t_{i-1} > 0 \) for \( 2 \leq i \leq n, \) and \( \tilde{f}^{(a)} \) is defined in (3.5). Furthermore,
\[
\begin{align*}
&g_n (dv, t_n | y^{(n)}) = \frac{h_n (dv, t_n | y^{(n)})}{\int_0^{(n)} h_n (dv, t_n | y^{(n)})}, \quad (3.9)
\end{align*}
\]

where \( u^{(n)} = (u_1, \ldots, u_n) \).

### 4. Numerical illustration

This section presents a numerical experiment associated with the CSP. Our test involves the computation of an inverse Laplace transform. For parsimony, we adopt the

†As in Bo et al. (2006), if \( \mu = 0 \) and \( \sigma = 1 \), the following power series \( I^{(a)}(x) \) on \([0, b)\) is a solution to equation (3.4):
\[
I^{(a)}(x) := g^{(a)}(x) - C^{(a)} h^{(a)}(x), \quad x \in [0, b],
\]

where
\[
\begin{align*}
g^{(a)}(x) &= \sum_{k=0}^{\infty} \frac{(2^{k}x)^{k-1}}{(2k)!} \prod_{j=0}^{k} (\lambda + 2j \alpha), \\
h^{(a)}(x) &= \sum_{k=0}^{\infty} \frac{2^{k}x^{2k+1}}{(2k+1)!} \prod_{j=0}^{k} (\lambda + (2j + 1) \alpha), \\
C^{(a)} &= \frac{\sum_{k=1}^{\infty} [2^{k}x^{2k-1}(2k - 1)!] \prod_{j=0}^{k-1} (\lambda + 2j \alpha)}{\sum_{k=0}^{\infty} [2^{k}x^{2k-1}(2k - 1)!] \prod_{j=0}^{k-1} (\lambda + (2j + 1) \alpha)^{2}}
\end{align*}
\]

with the convention \( \prod_{j=0}^{n} (\ldots) = 1 \). Expression (3.6) for \( I^{(a)} \) will be applied to our numerical experiment in section 4.
Table 1. Preference parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Drift coefficient, $\mu$</td>
<td>0</td>
</tr>
<tr>
<td>Decay coefficient, $\sigma$</td>
<td>1</td>
</tr>
<tr>
<td>Spot interest rate, $r$</td>
<td>0.06</td>
</tr>
<tr>
<td>Diffusion coefficient, $D$</td>
<td>1</td>
</tr>
<tr>
<td>Reflected upper bound, $b$</td>
<td>0.25</td>
</tr>
<tr>
<td>Initial asset value, $Q_0=x_0$</td>
<td>0.5</td>
</tr>
<tr>
<td>Added noisy source, $\xi_t$</td>
<td>$N(0,1)$</td>
</tr>
</tbody>
</table>

Preference parameters shown in table 1: Recall equation (2.1), which can be rewritten as

$$dQ_t = -Q_t dr + dW_t + dl_t - du_t,$$

$$Q_0 = x_0 = 0.5,$$

and the regulators $l$ and $u$ satisfy, for all $t>0$,

$$\int_0^t 1_{\{Q_s > 0\}} dl_s = 0, \quad \int_0^t 1_{\{Q_s < 1\}} du_s = 0.$$

According to theorem 3.5, the CSP $\ell(s, t)$ is given by

$$\ell(s, t) = \ell(s, t; Y_t), \quad \text{for } t > s,$$

and

$$\ell(s, t; y) = \int_{0.25}^1 \int_{y}^{\infty} \left[ L^{-1} \left( \frac{l^{(1)}(v)}{l^{(0)}(0.25)} \right) \right] (dr)$$

$$\times h(\frac{y}{\sqrt{25}h(du_s, y), \quad y \in \mathbb{R}}. \quad (4.2)$$

Here, $l^{(1)}$ is expressed by (3.6) and

$$\frac{h(\frac{y}{\sqrt{25}}, y, s)}{dy} \left[ (1/\sqrt{2\pi}) \exp\left(-\frac{(y-v)^2}{2} \right) \right] p(s, 0.5, v)$$

$$- \int_0^s \frac{p(s-r, 0.25, v)}{dy} \left[ L^{-1} \left( \frac{l^{(v)}(0.5)}{l^{(0)}(0.25)} \right) \right] (dr). \quad (4.3)$$

We first simulate the transition probability $p(t, 0.5, v)$ by virtue of lemma 3.2. Using the preference parameters in table 1, from lemma 3.2 we have

$$p(t, 0.5, v) = \pi(v) + m(v) \sum_{n=1}^{\infty} e^{-2^{n-1}v} \psi_n(0.5) \psi_n(v), \quad v \in [0, 1]. \quad (4.4)$$

where

$$\pi(v) = \frac{\sqrt{2} \phi(\sqrt{2})}{\Phi(\sqrt{2}) - \Phi(1)}, \quad m(v) = 2 e^{-v},$$

and the estimates of $\lambda_n$ and $\psi_n(\cdot)$ are given by

$$\lambda_n = \frac{\pi^2 n^2}{2} + \frac{2}{3} + O\left(\frac{1}{n^2}\right),$$

$$\psi_n(v) = \pm e^{v^2/2} \left[ \cos(n \pi v) + \frac{1}{n \pi} f(v) \sin(n \pi v) \right] + O\left(\frac{1}{n^2}\right),$$

$$f(v) = \frac{1}{6} v^3 - \frac{7}{6} v.$$

By convention, $\phi(\cdot)$ and $\Phi(\cdot)$ denote the standard normal density and the corresponding cumulative distribution function, respectively. Figure 1 displays the probability densities $p(t, 0.5, v)$ for three different time points $t=0.25, 0.125$ and 0.0625 and the stationary density $\pi = \pi(v)$. We ran a ‘GWR’ algorithm to calculate numerically the inverse Laplace transform $L^{-1}$. The left panel of figure 2 shows images of the inverse Laplace transforms $L^{-1}(l^{(0)}(0.25))/(l^{(1)}(0.25))(dr)$ with $u=0.45, 0.65$ and 1. The right panel shows the corresponding cumulative distribution functions of the inverse Laplace transforms.

It is convenient to compute the conditional survival function $\ell(s, t; y)$ using formula (4.2). From the 3σ principle for the Normal distribution, we can restrict $y$ to the interval $[-3, 4]$. The left panel of figure 3 shows the conditional survival functions $\ell(s, t; y)$ for five different points $t=2.5, 1.5, 0.5, -0.5$ and $-1.5$, and the right panel is the local display on $[0.05, 0.2] \times [0.3, 0.55]$. We find that the CSP is a decreasing function with respect to the time $t-s$ and the CSP increases as the observed value of the price $Y_s=y$ at time $s$ increases. These phenomena are consistent with intuition.

Finally, in table 2 we provide a comparison between the regulated case (model (4.1)) and the non-regulated
case (model (4.1) without the regulators $l$ and $u$). It shows that the CSP corresponding to the regulated case is smaller, which is also consistent with intuition (because of the effect of the regulator $u$, the likelihood of down-crossing the threshold level in the regulated case is larger than that in the non-regulated case). We also find that the effect of regulation seems to be more apparent when $t/C_0s$ is larger.

5. Concluding remarks

This paper reports the CDP in a regulated market using a structural approach. We use the reflected Ornstein–Uhlenbeck process with two-sided barriers to describe the price dynamics of the regulated goods or services under consideration. Default occurs due to the down-crossing of a certain threshold level by the price process. We assume that we can only observe the price process at some discrete time points (as partial information). In the respective cases of a single observation and multiple observations, the explicit expressions for the CSP are derived through the inverse Laplace transform. A numerical simulation associated with the CSP is presented. We also provide a comparison between the regulated case and the non-regulated case to demonstrate the effect of regulation. The CSP corresponding to the regulated case is found to be smaller than that in the non-regulated case. For calibration of the model, Bo et al. (2008) have presented a semi-parametric scheme for estimating the parameters.

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*Table 2. Conditional survival probabilities at $X_0 = y/C_0$ and 2.5 with $s=0.1$ (the non-regulated counterparts are given in parentheses).*

<table>
<thead>
<tr>
<th>$t - s$</th>
<th>$y = -1.5$</th>
<th>$y = 2.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.532 (0.541)</td>
<td>0.701 (0.698)</td>
</tr>
<tr>
<td>0.2</td>
<td>0.371 (0.386)</td>
<td>0.509 (0.528)</td>
</tr>
<tr>
<td>0.3</td>
<td>0.269 (0.300)</td>
<td>0.372 (0.421)</td>
</tr>
<tr>
<td>0.4</td>
<td>0.198 (0.242)</td>
<td>0.271 (0.345)</td>
</tr>
<tr>
<td>0.5</td>
<td>0.144 (0.201)</td>
<td>0.198 (0.288)</td>
</tr>
<tr>
<td>0.6</td>
<td>0.106 (0.169)</td>
<td>0.145 (0.244)</td>
</tr>
<tr>
<td>0.7</td>
<td>0.078 (0.144)</td>
<td>0.106 (0.208)</td>
</tr>
<tr>
<td>0.8</td>
<td>0.057 (0.123)</td>
<td>0.078 (0.179)</td>
</tr>
<tr>
<td>0.9</td>
<td>0.042 (0.106)</td>
<td>0.057 (0.154)</td>
</tr>
<tr>
<td>1.0</td>
<td>0.031 (0.092)</td>
<td>0.042 (0.134)</td>
</tr>
</tbody>
</table>

---

Footnotes:

*We apply proposition 2.1 of Allili et al. (2005) to calculate the CSPs via the inverse Laplace transform. Although the literature (Allili et al. 2005) presents three numerical methods for approaching the density of the first hitting time directly, the inverse Laplace transform seems to be more efficient than the others.*

*There is only one exception ($t(0.1, 0.2; 2.5)$), which may be due to calculation error.*
Appendix A: Proofs of the lemmas and theorems

Proof (proof of lemma 3.1): Recall (2.5). From lemma 2.1, it follows that, on the event \( \{ \tau > s \} \),
\[
\tilde{\ell}(s, t) = P(\tau > t \mid F_s) = P(\tau > t, \tau > s \mid F_s) = 1_{\{\tau > t\}} \frac{P(\tau > s \mid Y_{t-}, \ldots, Y_s)}{P(\tau > s \mid Y_{t-}, \ldots, Y_s)} = \begin{cases} 1, & \tau > t, Y_{t-}, \ldots, Y_s, \\ 0, & \text{otherwise} \end{cases}.
\]
where \( P(\tau > s \mid Y_{t-}, \ldots, Y_s) = 1 - \mathbb{E}(D_1 \mid \sigma(Y_{t-}, \ldots, Y_s)) \) denotes the conditional probability with respect to the \( \sigma \)-field \( \sigma(Y_{t-}, \ldots, Y_s) \). This completes the proof of the lemma. \( \square \)

Proof (proof of theorem 3.5): From the definition of the default time \( \tau \) in (2.2), it follows that
\[
P(\tau > t \mid Y_t) = P\left( \inf_{t \leq s < \tau} Q_s > d \mid Y_t \right)
= \int_0^\infty dP\left( \inf_{t \leq s < \tau} Q_s > d, \inf_{t \leq s < \tau} Q_s > d, Q_s \in dv \mid Y_t \right)
= \int_0^\infty \int_0^\infty \int_0^\infty \cdots \int_0^\infty dP\left( \inf_{t \leq s < \tau} Q_s > d \mid Y_{t-}, \tau > t \right) P(\tau > t \mid Y_t)
= \int_0^\infty dP\left( \inf_{t \leq s < \tau} Q_s > d \mid Y_{t-}, \tau > t \right) P(\tau > t \mid Y_t)
= \int_0^\infty \int_0^\infty \cdots \int_0^\infty dP\left( \inf_{t \leq s < \tau} Q_s > d \mid Q_{t-} = v, Y_{t-}, \tau > t \right) P(\tau > t \mid Y_t)
= \int_0^\infty \left[ \int_0^\infty \cdots \int_0^\infty \right] dP\left( Q_s \in dv \mid Y_{t-}, \tau > t \right) P(\tau > t \mid Y_t)
= \int_0^\infty \left[ \int_0^\infty \cdots \int_0^\infty \right] dP\left( Q_s \in dv \mid Y_{t-}, \tau > t \right) P(\tau > t \mid Y_t).
\]
where the last equality follows from the Markov property of \( Q \) and the independent property of \( Q \) and \( \xi \). Applying lemma 3.1, it follows that
\[
\tilde{\ell}(s, t) = \int_0^\infty dP_s(\tau > t - t_1) P(Q_{t-} \in dv \mid Y_{t-}, \tau > t_1).
\] (A1)

Define
\[
g(\hat{d}, y, t_1) = P(Q_{t-} \in dv \mid Y_{t-} \in dy, \tau > t_1),
\]
\[
h(\hat{d}, y, t_1) = P(Q_{t-} \in dv \mid \tau > t_1 \mid Y_{t-} \in dy).
\]

Then
\[
l(y, t_1) := P(\tau > t_1 \mid Y_{t-} \in dy) = \int_0^\infty h(\hat{d}, y, t_1).
\]
Furthermore,
\[
h(\hat{d}, y, t_1) = \frac{P(Q_{t-} \in dv, \tau > t_1 \mid Y_{t-} \in dy - v)}{P(Y_{t-} \in dy)} = \frac{P(Q_{t-} \in dv, \tau > t_1) P(\xi_{t-} \in dy - v)}{P(Y_{t-} \in dy)},
\] (A2)

and
\[
g(\hat{d}, y, t_1) = \frac{P(Q_{t-} \in dv, \tau > t_1 \mid Y_{t-} \in dy)}{l(y, t_1)} = \frac{h(\hat{d}, y, t_1)}{l(y, t_1)}.
\] (A3)

Observe (A1)-(A3). To obtain the CSP \( \tilde{\ell}(s, t) \), it suffices to calculate \( P(\tau > t_1, Q_{t-} \in dv) \). However, the argument
\[ P(\tau > t_1, Q_{n_1} \in dv) = P(Q_{n_1} \in dv) - P(Q_{n_1} \in dv, \tau \leq t_1) \]
\[ = P(Q_{n_1} \in dv) - \int_0^{t_1} P(Q_{n_1} \in dv | \tau) d\tau \]
\[ = \int_P \hat{p}(t, u, v) \mu(du)dv \]
\[ - \int_0^t P(Q_{n_1} \in dv | Q_u > d, 0 \leq u < s, Q_s = d) P(\tau \in ds) \]
\[ = \int_P \hat{p}(t, u, v) \mu(du)dv - \int_0^t P_d(Q_{n_1-s} \in dv) P(\tau \in ds) \]
\[ = \int_P \hat{p}(t, u, v) \mu(du)dv - \int_0^t P_d(Q_{n_1-s} \in dv) P(\tau \in ds) \]
\[ = \int_P \hat{p}(t, u, v) \mu(du)dv - \int_0^t \int_P \hat{p}(t - s, d, v) \left[ L^{-1}(\tilde{L}^{(i)}(d)) \right] (ds) \mu(du)dv. \] (A4)

Thus we have completed the proof of the theorem. \( \square \)

**Proof** (proof of theorem 3.6): From the results proved in subsection 3.1, it follows that, at observation time \( t_1 \),
\[ h_1(dy_{t_1}, t_1 | y_{t_1}) = \frac{P(Q_{t_1} \in dy_{t_1}, \tau > t_1) P(\xi_t \in dy_{t_1} - y_{t_1})}{P(Y_{t_1} \in dy_{t_1})}, \] (A5)
where the numerator on the right-hand side of (A5),
\[ P(Q_{t_1} \in dy_{t_1}, \tau > t_1) P(\xi_t \in dy_{t_1} - y_{t_1}), \]
is given in (A4). On the other hand, \[ h_d(dy^{(i)}, t_n | y^{(i)}) = \frac{P(Y^{(i)} \in dy^{(i)}, Y^{(i)} \in dy^{(i)}, \tau > t_n)}{P(Y^{(i)} \in dy^{(i)})}. \]

Consider the following three events:
\[ S_1 = \{ \tau > t_n, Q_{n_1} \in dv_{n_1}, Y_{n_1} \in dy_{n_1} \}, \]
\[ S_2 = \{ \tau > t_{n-1}, Y^{(n-1)} \in dy^{(n-1)}, Y^{(n-1)} \in dy^{(n-1)} \}, \] (A6)
\[ S_3 = \{ \tau > t_n, Y^{(n)} \in dy^{(n)}, Y^{(n)} \in dy^{(n)} \}. \]

Obviously, we have
\[ S_1 = S_1 \cap S_2. \]

Therefore,
\[ h_d(dy^{(n)}, t_n | y^{(n)}) = \frac{P(S_1, S_2)}{P(Y^{(n)} \in dy^{(n)})} \frac{P(S_1)}{P(Y^{(n)} \in dy^{(n)})} \frac{P(S_2)}{P(Y^{(n-1)} \in dy^{(n-1)})}. \] (A7)

Note that
\[ P(S_1 | S_2) = P(\tau > t_n, Q_{n_1} \in dv_{n_1}, \xi_t \in dy_{n_1} - y_{n_1} | S_2) \]
\[ = P(\tau > t_n, Q_{n_1} \in dv_{n_1} | \tau > t_{n-1}, Y^{(n-1)} \in dy^{(n-1)} \in dy^{(n-1)} \in dy^{(n-1)}) \times F_1(t_{n_1}, y_{n_1} - y_{n_1}). \] (A8)

From the strong Markov property and (A8), it follows that
\[ \frac{h_d(dy^{(n)}, t_n | y^{(n)})}{\frac{h_d(dy^{(n-1)}, t_{n-1} | y^{(n-1)})}{h_d(dy^{(n-1)}, t_{n-1} | y^{(n-1)})}} \]
\[ = \frac{P(y_{n_1}; \Delta t_{n_1}, y_{n_1} - y_{n_1}) - \frac{\Delta t_{n_1} \hat{p}(d, \Delta t_{n_1}, y_{n_1})}{L^{-1}(\tilde{L}^{(i)}(y_{n_1})) \tilde{L}^{(i)}(y_{n_1}))}}{\frac{\hat{p}(y_{n_1} | y^{(n-1)})}{\tilde{L}^{(i)}(y_{n_1}))}} \times F_1(y_{n_1} - y_{n_1}). \]

Thus we prove that (3.8) holds. By induction from (3.8), one can obtain the expression
\[ h_d(dy^{(i)}, t_i | y^{(i)}), \]
for \( 1 \leq i \leq n \). As for the conditional distribution \( g_d(dy^{(n)}, t_n | y^{(n)}) \) of \( Y^{(n)} \) conditioned on \( Y^{(n)} \), by virtue of Bayes’ rule, we have
\[ g_d(dy^{(n)}, t_n | y^{(n)}) = \frac{h_d(dy^{(n)}, t_n | y^{(n)})}{\int_{dy^{(n)}} h_d(dy^{(n)}, t_n | y^{(n)})}. \]

Thus the theorem is proved. \( \square \)