ON A NONLOCAL STOCHASTIC KURAMOTO–SIVASHINSKY EQUATION WITH JUMPS

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In this paper, we study a class of nonlocal stochastic Kuramoto–Sivashinsky equations driven by compensated Poisson random measures and show the existence and uniqueness of the weak solution to the equation. Furthermore, we prove that an invariant measure of the equation indeed exists under some appropriate assumptions.

Keywords: Nonlocal Kuramoto–Sivashinsky equation; Poisson random measure; invariant measure.

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1. Introduction

Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) be some complete filtered probability space satisfying the usual condition, and on which, \(N(dz, dt) := N(dz, dt) - \Pi(dz)dt\) defines a compensated Poisson random measure of a Poisson random measure \(N: \mathcal{B}(Z) \times \mathbb{R}_+ \times \Omega \to \mathbb{N} \cup \{0\}\) with the characteristic measure \(\Pi\) on some measurable space \((Z, \mathcal{B}(Z))\) satisfying \(\Pi(Z) < \infty\). From [6], it follows that \(\{N((0,t] \times A); (t, A) \in \mathbb{R}_+ \times \mathcal{B}(Z)\}\) can be represented by a \(Z\)-valued point function \(\{p(t); t \geq 0\}\) with the domain \(D_p\) as a countable subset of \(\mathbb{R}_+\). That is,

\[
N((0,t] \times A) = \sum_{s \in D_p, s \leq t} 1_A(p(s)), \quad \text{for } t > 0 \quad \text{and} \quad A \in \mathcal{B}(Z).
\]

We are concerned with the following one-dimensional nonlocal stochastic Kuramoto–Sivashinsky (K–S) equation driven by a compensated Poisson random
measure with periodic boundary conditions, which is described by

\[
\begin{aligned}
\frac{\partial X(\xi,t)}{\partial t} + \frac{\partial^4 X(\xi,t)}{\partial \xi^4} + \frac{\partial^2 X(\xi,t)}{\partial \xi^2} + X(\xi,t) \frac{\partial X(\xi,t)}{\partial \xi} + \alpha \mathcal{H} \left( \frac{\partial^3 X(\xi,t)}{\partial \xi^3} \right) = \\
\int_Z \sigma(X(\xi,t-), z) \tilde{N}(dz, dt);
\end{aligned}
\]

(1.1)

where \(\alpha\) and \(L\) are some positive constants, and the nonlocal term \(\mathcal{H}(\cdot)\) is the Hilbert transform admitting the following form:

\[
\mathcal{H}(u)(\xi) = \frac{-1}{2L} \int_G \cot \left( \frac{\xi - \eta}{2L} \right) u(\eta) d\eta, \quad \xi \in G.
\]

(1.2)

As is well known, the deterministic K-S equation (which corresponds to the case \(\sigma \equiv 0\) and \(\alpha = 0\) in (1.1)) arises in the modelling of the flow of a thin film of viscous liquid falling down an inclined plane, subject to an applied electric field. Under the impact of a nonlocal term, Duan and Vincent [4] studied the dynamics concerning deterministic nonlocal K-S equation. In a successive paper [5], the authors discussed a stochastic version of the equation with an additive white noise, but without the nonlocal term (i.e. \(\alpha = 0\)). They proved that a unique weak solution exists in \(L^4(0, T; L^4(G))\), \(\mathbb{P}\)-a.s. for the equation with homogeneous Dirichlet boundary conditions. In [11], Yang studied the analogous subject as in [4] for the equation driven by an additive white noise, under the impact of the nonlocal term \(\mathcal{H}(\cdot)\). On the other hand, a recent paper by Dong and Xu [3] shows the existence and uniqueness of the weak (strong) solution for one-dimensional stochastic Burgers equations driven by compensated Poisson random measures instead of white noises. We also notice that Bardu and Da Prato [1] studied the ergodicity of variational solutions of stochastic partial differential equations (SPDEs) with Gaussian white noises. In particular, the variation solution of the SPDE was first proposed by Pardoux [9]. Returning to Eq. (1.1), our present goal is to prove the existence of an invariant measure for (1.1), based on the existence and uniqueness of the weak solution of the equation.

The rest of this paper is organized as follows: In Sec. 2, some preliminaries and main hypothesis are given. The existence and uniqueness of the global weak solution to (1.1) are established in Sec. 3. Section 4 is devoted to proving an invariant measure of (1.1) exists under some appropriate assumptions.

Throughout the paper, the generic positive constant \(C\) may change from line to line.

2. Preliminaries and Hypothesis

We begin with some basic notations, functional spaces and inequalities, which will be used frequently in the following sections. At first, since the solution of (1.1) with
For each $A$, let $\sigma \equiv 0$ is periodic, for all $t \geq 0$,

$$\bar{x} := \frac{1}{2L} \int_{-L}^{L} X(t, \xi) d\xi = \frac{1}{2L} \int_{-L}^{L} x(\xi) d\xi.$$  

Without loss of generality, we assume that $\bar{x} = 0$, and define:

$$H := \left\{ u \in L^2(G) : u \text{ is periodic on } G, \int_{-L}^{L} u(x) dx = 0 \right\},$$

$$H^p_{\text{per}} := \left\{ u \in W^{2,p}(G) : u \text{ is periodic on } G \right\}, \text{ for each } p \in \mathbb{N},$$

$$\tilde{H}^p_{\text{per}} := H^p_{\text{per}} \cap H, \text{ for each } p \in \mathbb{N}.$$  

For each $i \in \mathbb{N} \cup \{0\}$, set $D_i := \frac{\partial^i}{\partial x^i}$ and $D_0 = I$ (the identity operator on $H)$. Let $A = -D_2$ and $V = D(A)$, then $A$ is a positive self-adjoint unbounded linear operator on $V$. Set

$$\lambda_k = \pi^2 k^2 \frac{L}{a^2}, \quad \phi_k(x) = \frac{1}{\sqrt{L}} \sin \left( \frac{k\pi x}{L} \right), \quad \psi_k(x) = \frac{1}{\sqrt{L}} \cos \left( \frac{k\pi x}{L} \right),$$

then $(\lambda_k)_{k \in \mathbb{N}}$ and $(\psi_k)_{k \in \mathbb{N}}$ be the eigenvalues and the corresponding eigenfunctions of $A : V \rightarrow H$, and $(\psi_k)_{k \in \mathbb{N}}$ forms a complete orthonormal basis of $H$. The spectral theory of operator $A$ allows us to define the powers $A^s$ of $A$ by (see e.g. [2]), for $s \in \mathbb{R},$

$$D(A^s) = \left\{ u \in H : \sum_{k=1}^{\infty} \lambda_k^s (u, e_k)^2 < \infty \right\},$$

$$A^s u = \sum_{k=1}^{\infty} \lambda_k^s (u, e_k) e_k, \quad \text{for } u \in D(A^s),$$

where $(\cdot, \cdot)$ and $| \cdot |$ respectively denote the inner product and the corresponding norm of $H$. We endow the domain $D(A^s)$ of $A^s : D(A^s) \rightarrow H$ with the following inner product and the norm:

$$\left\{ \begin{array}{l}
(u, v)_{2s} = (A^s u, A^s v), \\
|u|_{2s} = |A^s u|
\end{array} \right.$$  

for $u, v \in D(A^s)$. Obviously, we have,

$$D(A^0) = H, \quad D(A^{1/2}) = \tilde{H}^1_{\text{per}} \quad \text{and} \quad V = \tilde{H}^2_{\text{per}}.$$  

The following inequalities are well known (see e.g. [10, 12]):

Poincaré type inequality: $|u|_\alpha \leq \frac{\lambda_1^{\alpha-\beta}}{\lambda_1^{\beta-\alpha}} |u|_\beta$, for $\alpha \leq \beta$, $u \in D(A^{\beta/2})$, \hfill (2.1)

Interpolation inequality: $|u|_\beta \leq |u|_\alpha^{\frac{\alpha-\beta}{\alpha}} |u|_\gamma^{\frac{\beta-\gamma}{\alpha}}$, for $\alpha < \beta < \gamma$, $u \in D(A^{\gamma/2})$, \hfill (2.2)

Agmon estimate: $|u|_{L^\infty(G)} \leq |u|_{2s}^{\frac{1}{2}} |u|_{s}^{\frac{1}{2}}$, for $u \in D(A^{1/2})$. \hfill (2.3)
For $u \in \tilde{H}_k^1$, $|D_k(u)|$ is an equivalent norm in $\tilde{H}_k^1$ from (2.1). For the Burger’s term $uD_1u$, we define a trilinear form as in [2] by
\[
b(u, v, z) = \int_{-L}^{L} u(\xi) (D_1v)(\xi)z(\xi)d\xi.
\]
Define a bilinear continuous operator $B : D(A^{1/2}) \times D(A^{1/2}) \rightarrow H$ by $(B(u, v), z) = b(u, v, z)$ for $u, v, z \in D(A^{1/2})$. From the periodicity of the boundary condition, it follows that $(B(u, u), u) = 0$. Then (1.1) might be rewritten as the following abstract form:
\[
\begin{align*}
dX(t) + (A^2X(t) - AX(t) + B(X(t)) + \alpha H((-A)^{3/2}X(t)))dt &= \\
&= \int_Z \sigma(X(t-), z)\tilde{N}(dz, dt), \\
X(0) &= x.
\end{align*}
\]
(2.4)

Hereafter, we will study (2.4) instead of (1.1). As for the nonlocal term $H(\cdot)$ in (1.1), but defined by (1.2), it has the following properties (see e.g. [4]):
\[
\begin{align*}
D_1H(u) &= H(D_1u), \\
(v, H(u)) &= -(u, H(v)), \\
(H(u), H(v)) &= (u, v), \\
|H(u)| &= |u|.
\end{align*}
\]
(2.5)

Next we state a lemma, which is used for proving the existence of the weak solution of (2.4), but we omit the proof of the lemma (see e.g. [10]).

**Lemma 2.1.** Let $E_1 \subseteq E_2 \subseteq E_3$ be Banach spaces with continuous embedding. Suppose that $\{X_m\}_{m=1}^\infty$ and $\{dX_m/dt\}_{m=1}^\infty$ are bounded in $L^2(0, T; E_1)$ and $L^p(0, T; E_3)$ for $p > 1$, respectively. If $E_1$ is reflexive space and the embedding $E_1 \subseteq E_2$ is compact, then there exists a subsequence $\{X_{m_k}\}_{k=1}^\infty$ of $\{X_m\}_{m=1}^\infty$, which converges in $L^2(0, T; E_2)$.

At the end of the section, we make two assumptions on Eq. (1.1) or (2.4):

(H1) Assume that there exists a constant $\ell > 0$ such that
\[
\int_Z |\sigma(x, z) - \sigma(y, z)|^2\rho(dz) \leq \ell|x - y|^2, \quad \forall x, y \in H.
\]
(H2) Recall the eigenvalue $\lambda_1 = \frac{\pi^2}{L^2}$ and the Lipschitzian coefficient $\ell$ in (H1). Assume that,
\[
\gamma := \lambda_1^2 - (2(1 + \alpha^2)^2 + \ell^2) > 0.
\]
(H3) For every $(x, z) \in H \times Z$, $\sigma(x, z)$ is bounded in $H$. 
Remark 2.1. It is easy to check that there are many examples of SPDEs with coefficients satisfying (H1)–(H3).

3. Existence and Uniqueness

In this section, we shall establish the existence and uniqueness of the weak solution to (2.4). We begin with our main theorem in this section.

**Theorem 3.1.** Under the assumption (H1), for initial $x \in H$, Eq. (2.4) admits a unique RCLL (right-continuous with left-limit) version of global weak solution in $L^\infty(0,T;H) \cap L^2(0,T;V)$, $P$-a.s., for any $T > 0$.

First of all, we consider the following deterministic PDE,

$$
\begin{cases}
    dY(t) + (A^2Y(t) - AY(t) + B(Y(t)) + \alpha \mathcal{H}((-A)^{3/2}Y(t)))dt \\
    = -\int_Z \sigma(Y(t), z) \Pi(dz)dt, \\
    Y(0) = x \in H,
\end{cases}
$$

and we have the following conclusion:

**Proposition 3.1.** Under the same conditions of Theorem 3.1, there exists a unique global weak solution $Y := \{Y(t); t \geq 0\}$ of (3.1) such that $Y \in C(0,T;H) \cap L^2(0,T;V)$, for any $T > 0$.

To prove the proposition, we mainly adopt the Galerkin method. Let $H_m = \text{span}\{e_1, e_2, \ldots, e_m\}$, and $P_m$ be the orthogonal projection operator from $H$ onto $H_m$. Then the Galerkin equations associated with Eq. (3.1) is

$$
\begin{cases}
    \frac{dY_m(t)}{dt} = -A^2Y_m(t) + AY_m(t) - P_mB(Y_m(t)) \\
    - \alpha \mathcal{H}((-A)^{3/2}Y_m(t)) - P_m \int_Z \sigma(Y_m(t), z) \Pi(dz), \\
    Y_m(0) = P_m x.
\end{cases}
$$

We first treat Eq. (3.2), and we have

**Lemma 3.1.** For each $T > 0$ and $m = 1, 2, \ldots$, there exists a unique solution $Y_m \in C(0,T;H) \cap L^2(0,T;V)$ of (3.2), provided the same conditions of Theorem 3.1 hold.

**Proof.** Since Eq. (3.2) is an $m$-dimensional ordinary differential equation (ODE) in $H_m$. It is not difficult to obtain the existence of a solution in $C(0,T;H) \cap L^2(0,T;V)$ on some time interval $[0, T_m]$ by a standard argument (see e.g. [10]). However, we have to use the Agmon estimate (2.3) in this argument due to the impact of the
nonlocal term $\mathcal{H}(\cdot)$. From (3.2), note that, for $m = 1, 2, \ldots$,
\[
\frac{1}{2} \frac{d}{dt} |Y_m(t)|^2 + |D_2Y_m(t)|^2 = |D_1Y_m(t)|^2 + \alpha \int_G Y_m(t, \xi) \mathcal{H}(D_3Y_m)(t, \xi) d\xi - \left( P_m \int_Z \sigma(Y_m(t, \xi) \Pi(d\xi), Y_m(t) \right).
\]

From (2.5), it follows that
\[
\left| \alpha \int_G Y_m(t, \xi) \mathcal{H}(D_3Y_m)(t, \xi) d\xi \right| \leq \alpha |D_1Y_m(t)||D_2Y_m(t)| \leq \frac{1}{4} |D_2Y_m(t)|^2 + \alpha^2|D_1Y_m(t)|^2.
\]

By the interpolation inequality (2.2) with $\beta = 1, \alpha = 0$ and $\gamma = 2$,
\[
2(1 + \alpha^2)|D_1Y_m(t)|^2 \leq 2(1 + \alpha^2)|Y_m(t)||D_2Y_m(t)| \leq \frac{1}{2} |D_2Y_m(t)|^2 + 2(1 + \alpha^2)|Y_m(t)|^2.
\]

On the other hand, assumption (H1) implies that,
\[
\left| \int_G \sigma(Y_m(t, \xi) \Pi(d\xi), Y_m(t) \right| \leq C \left( \sqrt{\Pi(Z)} + \Pi(Z) \right) |Y_m(t)|^2 + C^*.
\]

where $C^* := \int_Z |\sigma(0, z)|^2 \Pi(dz)$. So that the estimates (3.3)–(3.5) jointly imply that,
\[
\frac{d}{dt} |Y_m(t)|^2 + |D_2Y_m(t)|^2 \leq \left[ 2(1 + \alpha^2)^2 + C \left( \sqrt{\Pi(Z)} + \Pi(Z) \right) \right] |Y_m(t)|^2 + C^*.
\]

Therefore for all $t > 0$,
\[
|Y_m(t)|^2 + \int_0^t |Y_m(s)|^2 ds \leq (|x|^2 + C^*t) + \left[ 2(1 + \alpha^2)^2 + C \left( \sqrt{\Pi(Z)} + \Pi(Z) \right) \right] \int_0^t |Y_m(s)|^2 ds.
\]

Thus the Gronwall’s lemma yields that, for all $t \in [0, T]$,
\[
|Y_m(t)|^2 \leq (|x|^2 + C^*T) \exp \left\{ \left[ 2(1 + \alpha^2)^2 + C \left( \sqrt{\Pi(Z)} + \Pi(Z) \right) \right] T \right\}.
\]

Note that $\Pi(Z) < \infty$. This shows that the local solution does not diverge at finite time, i.e. $T_m = T$. Thus we complete the proof of the lemma.
Lemma 3.2. Let $Z \in C(0,T;H) \cap L^2(0,T;V)$ for any $T > 0$. Then

\begin{enumerate}
  \item \[
    \int_0^T |D_2 Z(t)|^2_{V^*} \, dt \leq |Z|_{L^2(0,T;H)}^2,
  \]
  \item \[
    \int_0^T |D_4 Z(t)|^2_{V^*} \, dt \leq |Z|_{L^2(0,T;V)}^2,
  \]
  \item \[
    \int_0^T |B(Z(t))|^2_{V^*} \, dt \leq C |Z|_{L^\infty(0,T;H)}^2 |Z|_{L^2(0,T;V)}^2,
  \]
  \item \[
    \int_0^T |\mathcal{H}(D_3 Z(t))|^2_{V^*} \, dt \leq C |Z|_{L^2(0,T;V)}^2,
  \]
  \item \[
    \int_0^T \left| \int_Z \sigma(Z(t),z) \Pi(dz) \right|^2_{V^*} \, dt \leq 2TC_* \Pi(Z) \left( 1 + |Z|_{L^2(0,T;H)}^2 \right),
  \]
\end{enumerate}

where $C_* := \max\{\ell, \int_Z |\sigma(0,z)|^2 \Pi(dz)\}$.

Proof. (1) By the definition of $|\cdot|_{V^*}$, we have,

\[
\int_0^T |D_2 Z(t)|^2_{V^*} \, dt = \int_0^T \sup_{|U|_2 = 1} |(Z(t), D_2 U)|^2 \, dt \\
\leq \int_0^T |Z(t)|^2 \, dt \\
= |Z|_{L^2(0,T;H)}^2.
\]

The proof of (2) is similar to that of (1). We omit its proof.

(3) From Lemma 2.2 in [3], it follows that there exists a positive constant $C > 0$ such that

\[
\int_0^T |B(Z(t))|^2_{V^*} \, dt = \int_0^T \sup_{|U|_2 = 1} |b(Z(t), Z(t), U)|^2 \, dt \\
\leq C \int_0^T |Z(t)|^2 |Z(t)|_2^2 \, dt \\
= C |Z|_{L^\infty(0,T;H)}^2 |Z|_{L^2(0,T;V)}^2.
\]

(4) By the Poincaré type inequality (2.1) and the property (2.5),

\[
\int_0^T |\mathcal{H}(D_3 Z(t))|^2_{V^*} \, dt = \int_0^T \sup_{|U|_2 = 1} |(\mathcal{H}(D_1 Z(t)), D_2 U)|^2 \, dt \\
\leq \int_0^T |\mathcal{H}(D_1 Z(t))|^2 \, dt.
\]
for some constant $C > 0$. As for the proof of (5), note that for $U \in H$, $|U|_{V^*}$, \leq |U|$, then

$$
\int_0^T \left| \int_Z \sigma(Z(t), z)\Pi(dz) \right|^2_{V^*} dt
= \int_0^T \left| \int_Z \sigma(Z(t) - Z(0), z)\Pi(dz) + \int_Z \sigma(Z(0), z)\Pi(dz) \right|^2_{V^*} dt
\leq 2 \int_0^T \left| \int_Z \sigma(Z(t) - Z(0), z)\Pi(dz) \right|^2_{V^*} dt
+ 2 \int_0^T \left| \int_Z \sigma(Z(0), z)\Pi(dz) \right|^2_{V^*} dt
\leq 2TC_\Pi(Z) \left( 1 + \left| Z \right|_{L^2(Z;H)}^2 \right).
$$

Thus the proof of the lemma is complete. \hfill \Box

According to the proof of Lemmas 3.1 and 3.2, we might obtain the following facts:
(a) $\{Y_m(t)\}_{m \geq 1}$ is bounded in $L^2(0, T; V) \cap L^\infty(0, T; H)$, and for any $T > 0$,
the following estimate holds,

$$
\int_0^T |Y_m(t)|_2^2 dt \leq (|x|^2 + C^*T) \exp \left\{ \left[ 2(1 + \alpha^2)^2 + C \left( \sqrt{\Pi(Z)} + \Pi(Z) \right) \right] T \right\}.
$$

(b) $\left\{ \frac{d}{dt} Y_m(t) \right\}_{m \geq 1}$ is bounded in $L^2(0, T; V^*)$. In fact, for any $T > 0$,

$$
\int_0^T \left| \frac{d}{dt} Y_m(t) \right|^2_{V^*} dt
\leq 2 \int_0^T |D_4 Y_m(t)|_{V^*}^2 dt + 2 \int_0^T |D_2 Y_m(t)|_{V^*}^2 dt
+ 2 \int_0^T |B(Y_m(t))|_{V^*}^2 dt + 2\alpha \int_0^T |H(D_3 Y_m(t))|_{V^*}^2 dt
+ 2 \int_0^T \left| \int_Z \sigma(Y_m(t), z)\Pi(dz) \right|^2_{V^*} dt.
$$
\[\leq 2|Y_m|_{L^2(0,T;V)}^2 + 2|Y_m|_{L^\infty(0,T;H)}^2 + 2C|Y_m|_{L^\infty(0,T;H)}^2 |Y_m|_{L^2(0,T;V)}^2 + 2TC_\pi(Z)\left[1 + |Y_m|_{L^\infty(0,T;H)}^2\right].\]

Then the energy estimates (a)–(b) imply that, there exists a subsequence of \(\{Y_m\}_{m=1}^\infty\), which is also indexed by \(\{Y_m\}_{m=1}^\infty\) such that as \(m \to \infty\),

\[
\begin{align*}
Y_m &\to Y, \quad \text{in } L^2(0,T;V), \\
Y_m &\rightharpoonup Y, \quad \text{in } L^\infty(0,T;H), \\
\frac{d}{dt}Y_m &\rightharpoonup \dot{Y}, \quad \text{in } L^2(0,T;V^*).
\end{align*}
\]

(3.8)

Furthermore, we also have

**Lemma 3.3.** Under the same conditions of Theorem 3.1, we have,

(1) \(Y = \dot{Y}\), in \(L^\infty(0,T;H)\). (2) \(\frac{d}{dt}Y = \dot{Y}\), in \(L^2(0,T;V^*)\).

**Proof.** The proof of the lemma is somehow standard (see e.g. [3, 10]). We only prove (1), since the proof of (2) is similar. Let \(U \in L^2(0,T;H)\). Then, by (3.8), as \(m \to \infty\),

\[
\int_0^T (Y_m(t),U(t))dt \to \int_0^T (\dot{Y}(t),U(t))dt.
\]

Note that the embedding \(H^1_{\text{per}} \subset V\) is compact. It follows from Lemma 2.1 and the fact (a) that, there exists a subsequence of \(\{Y_m\}_{m=1}^\infty\), yet denoted by \(\{Y_m\}_{m=1}^\infty\) such that \(Y_m \to Y\) in \(L^2(0,T;H^1_{\text{per}})\) for \(m \to \infty\). Since \(U \in L^2(0,T;H)\), then there exists an element \(\tilde{U} \in L^2(0,T;V)\) such that \(D_2\tilde{U}(t) = U(t)\). Therefore,

\[
\int_0^T (Y_m(t),U(t))dt = \int_0^T (Y_m(t),\tilde{U}(t))_1 dt \\
\to \int_0^T (Y(t),\tilde{U}(t))_1 dt = \int_0^T (Y(t),U(t))dt,
\]

as \(m \to \infty\). Thus for arbitrary \(U \in L^2(0,T;H)\),

\[
\int_0^T (Y(t) - \dot{Y}(t),U(t))dt = 0.
\]

Take \(U(t) = Y(t) - \dot{Y}(t)\). Then \(Y(t) = \dot{Y}(t)\) in \(H\), for all \(t \in [0,T]\). In particular, \(Y = \dot{Y}\) in \(L^\infty(0,T;H)\). Thus the proof of the lemma is completed. \(\square\)
Thanks to Lemma 3.2, there exists a $Y \in L^\infty(0, T; H) \cap L^2(0, T; V)$ such that as $m \to \infty$,

$$\begin{align*}
Y_m \to Y, & \quad \text{in } L^2(0, T; V), \\
Y_m \overset{*}{\to} Y, & \quad \text{in } L^\infty(0, T; H), \\
\frac{d}{dt} Y_m \overset{*}{\to} \frac{d}{dt} Y, & \quad \text{in } L^2(0, T; V^*). 
\end{align*}$$

(3.9)

Next we check that the $Y$ in (3.9) satisfies Eq. (3.2).

**Proof of Proposition 3.1.** We will proceed with the proof into four steps:

**Step 1.** To check $D_1 Y_m \to D_1 Y$ in $L^2(0, T; V^*)$ and $D_2 Y_m \to D_2 Y$ in $L^2(0, T; V^*)$.

**Step 2.** To check $P_m B(Y_m) \to B(Y)$ in $L^2(0, T; V^*)$, as $m \to \infty$.

**Step 3.** To check $P_m \int_Z \sigma(Y_m(t), z)|\Pi(dz) \to \int_Z \sigma(Y(t), z)|\Pi(dz)$ in $L^2(0, T; V^*)$, as $m \to \infty$.

**Step 4.** To check $P_m \mathcal{H}(D_3 Y_m) \to \mathcal{H}(D_3 Y)$ in $L^2(0, T; V^*)$, as $m \to \infty$. The proofs of Steps 1–3 are similar to those in [3]. We only need to check Step 4. Since, for $U \in C(0, T; V)$,

$$\int_0^T |(P_m \mathcal{H}(D_3 Y_m(t)) - \mathcal{H}(D_3 Y(t)), U(t))| \, dt$$

$$\leq \int_0^T |(\mathcal{H}(D_3 Y_m), (P_m - I)U(t))| \, dt$$

$$+ \int_0^T |(\mathcal{H}(D_3 Y_m(t)) - \mathcal{H}(D_3 Y(t)), U(t))| \, dt$$

$$:= J_1(Y) + J_2(Y).$$

(3.10)

If we define

$$\mathcal{U} := \left\{ \sum_{i=1}^n a_i(t)U_i : U_i \in V, a_i \in C([0, T]), n \geq 1 \right\},$$

and let $U \in \mathcal{U}$. Then for $n = 1, 2, \ldots$,

$$J_1(Y) \leq \int_0^T |\mathcal{H}(D_3 Y_m(t))|_{V^*} |(P_m - I)U(t)|_2 \, dt$$

$$\leq \sum_{i=1}^n \sup_{t \in [0, T]} |a_i(t)| |(P_m - I)U_i|_2 \int_0^T |\mathcal{H}(D_3 Y_m(t))|_{V^*} \, dt$$

$$\leq \sum_{i=1}^n \sup_{t \in [0, T]} |a_i(t)| |(P_m - I)U_i|_2 \int_0^T |\mathcal{H}(Y_m(t))|_1 \, dt.$$
\[
\begin{align*}
\leq & \sum_{i=1}^{n} \sup_{t \in [0,T]} |a_i(t)||\langle P_m - I \rangle U_i|_2 \int_{0}^{T} |Y_m(t)|^2_2 dt \\
\leq & C \sqrt{T} \sum_{i=1}^{n} \sup_{t \in [0,T]} |a_i(t)||\langle P_m - I \rangle U_i|_2 \left( \int_{0}^{T} |Y_m(t)|^2_2 dt \right)^{1/2} \\
\leq & C \sup_{m \in \mathbb{N}} |Y_m|_{L^2(0,T;V)} \sum_{i=1}^{n} \sup_{t \in [0,T]} |a_i(t)||\langle P_m - I \rangle U_i|_2 \\
\to 0, \quad & \text{as } m \to \infty. \quad (3.11)
\end{align*}
\]

On the other hand, by the proof of (4) in Lemma 3.2,

\[
\begin{align*}
J_2(Y) & \leq \int_{0}^{T} |U(t)|^2_2 |\mathcal{H}(D^3(Y_m(t) - Y(t)))|_V \cdot dt \\
& \leq C \left( \int_{0}^{T} |U(t)|^2_2 dt \right)^{1/2} \left( \int_{0}^{T} |\mathcal{H}(D^3(Y_m(t) - Y(t)))|^2_V \cdot dt \right)^{1/2} \\
& \leq C |U|_{L^2(0,T;V)} \left( \int_{0}^{T} |Y_m(t) - Y(t)|^2_2 dt \right)^{1/2} \\
& \to 0, \quad \text{as } m \to \infty. \quad (3.12)
\end{align*}
\]

The conclusion of Step 4 follows from the fact that \( \mathcal{U} \) is dense in \( C(0,T;V) \).

Next, we prove that the weak limit point \( Y \) given in (3.9) is a weak solution of (3.2). Take the weak limit in \( L^2(0,T;V^*) \) as \( m \to \infty \) on both sides of (3.2). Then by Steps 1–4,

\[
\frac{dY(t)}{dt} = -A^2 Y(t) + AY(t) - B(Y(t)) - \alpha \mathcal{H}((-A)^{3/2} Y(t)) \\
- \int_{Z} \sigma(Y(t),z) \Pi(dz) \quad (3.13)
\]

holds in \( L^2(0,T,V^*) \). We check \( Y(0) = x \). Let \( \mathcal{S} \in C^1(0,T;V) \) and \( \mathcal{S}(T) = 0 \).

By (3.13),

\[
\begin{align*}
- \int_{0}^{T} \langle Y(t), \mathcal{S}'(t) \rangle dt &= (Y(0),\mathcal{S}(0)) - \int_{0}^{T} \langle A^2 Y(t), \mathcal{S}(t) \rangle dt + \int_{0}^{T} \langle AY(t), \mathcal{S}(t) \rangle dt \\
& \quad - \int_{0}^{T} \langle B(Y(t)), \mathcal{S}(t) \rangle dt - \alpha \int_{0}^{T} \langle \mathcal{H}(A^{3/2} Y(t)), \mathcal{S}(t) \rangle dt \\
& \quad - \int_{0}^{T} \int_{Z} \langle \sigma(Y(t),z) \mathcal{S}(t) \rangle \Pi(dz) dt. \quad (3.14)
\end{align*}
\]
So that (3.14) and (3.15) imply that
\[
\int_0^T (Y(t), S'(t)) dt
\]
\[
= (x, S(0)) - \int_0^T \langle A^2 Y(t), S(t) \rangle dt + \int_0^T \langle AY(t), S(t) \rangle dt
\]
\[
- \int_0^T (B(Y(t)), S(t)) dt - \alpha \int_0^T (H(A^{3/2} Y(t)), S(t)) dt
\]
\[
- \int_0^T \int_Z (\sigma(Y(t), z) S(t)) \Pi(dz) dt.
\]
(3.15)

So that (3.14) and (3.15) imply that \( Y(0) = x \). Recall (3.9). Then \( Y \in L^2(0, T; V) \) and \( \frac{dY}{dt} \in L^2(0, T; V^*) \). Hence, by Corollary 7.3 in [10], we obtain \( Y \in C(0, T; H) \).

Finally, we prove the uniqueness of the solution. Let \( Y^1 \) and \( Y^2 \) be two global weak solutions of (3.2) with the same initial value. Let \( \Phi(t) = Y^{(1)}(t) - Y^{(2)}(t) \). Then in \( V^* \),
\[
\frac{d\Phi(t)}{dt} + A^2 \Phi(t) - A \Phi(t) + (B(Y^1(t)) - B(Y^2(t))) + \alpha H \left( -A \right)^{3/2} \Phi(t)
\]
\[
= -\int_Z \left( \sigma(Y^{(1)}(t), z) - \sigma(Y^{(2)}(t), z) \right) \Pi(dz), \quad 0 \leq t \leq T.
\]
(3.16)

Take the inner product with \( \Phi(t) \) in \( H \) on both sides of (3.16),
\[
\frac{1}{2} \frac{d}{dt} |\Phi(t)|^2 + |D_2 \Phi(t)|^2 - |D_1 \Phi(t)|^2 + \left( \left( B(Y^{(1)}(t)) - B(Y^{(2)}(t)) \right), \Phi(t) \right)
\]
\[
+ \alpha \left( H \left( -A \right)^{3/2} \Phi(t), \Phi(t) \right)
\]
\[
= \left( -\int_Z \left( \sigma(Y^{(1)}(t), z) - \sigma(Y^{(2)}(t), z) \right), \Phi(t) \right) \Pi(dz), \quad 0 \leq t \leq T.
\]
(3.17)

Then, by the periodic boundary condition, for \( t \in [0, T] \), we have
\[
\left( \left( B(Y^{(1)}(t)) - B(Y^{(2)}(t)) \right), \Phi(t) \right)
\]
\[
= \left( Y^{(1)}(t) D_1 Y^{(1)}(t) - Y^{(2)}(t) D_1 Y^{(2)}(t), \Phi(t) \right)
\]
\[
= \left( Y^{(1)}(t) D_1 \Phi(t) + \Phi(t) D_1 Y^{(2)}(t), \Phi(t) \right)
\]
\[
= \left( \Phi(t) D_1 \Phi(t), Y^{(1)}(t) \right) + \left( \Phi^2(t), D_1 Y^{(2)}(t) \right)
\]
\[
= \left( \Phi(t) D_1 \Phi(t), Y^{(1)}(t) \right) - 2 \left( \Phi(t) D_1 \Phi(t), Y^{(2)}(t) \right)
\]
\[
= (\Phi(t) D_1 \Phi(t), \Phi(t)) - \left( \Phi(t) D_1 \Phi(t), Y^{(2)}(t) \right)
\]
\[
= - \left( \Phi(t) D_1 \Phi(t), Y^{(2)}(t) \right),
\]
where we used the fact that \( (\Phi(t)D_1\Phi(t), \Phi(t)) = 0 \), for \( t \geq 0 \). By the Agmon estimate (2.3) and interpolation inequality (2.2) with \( \beta = 1, \alpha = 0 \) and \( \gamma = 2 \), one can obtain

\[
\left| \left( B(Y^{(1)}(t)) - B(Y^{(2)}(t)) \right), \Phi(t) \right| \\
\leq \sup_{\xi \in G} |\Phi(t, \xi)| |D_1\Phi(t)| |Y^{(2)}(t)| \\
\leq |D_1\Phi(t)|^2 |Y^{(2)}(t)| \\
\leq |\Phi(t)| |D_2\Phi(t)| |Y^{(2)}(t)| \\
\leq 2|Y^{(2)}(t)|^2 |\Phi(t)|^2 + \frac{1}{8} |D_2\Phi(t)|^2. \tag{3.18}
\]

It follows from assumption (H1) that, for \( t \in [0, T] \),

\[
\left| \left( \int_Z \left( \sigma(Y^{(1)}(t), z) - \sigma(Y^{(2)}(t), z) \right) \Pi(dz), \Phi(t) \right) \right| \\
\leq \sqrt{\Pi(Z)} |\Phi(t)|^2. \tag{3.19}
\]

Similarly, as in the proof of (3.3)–(3.4), together with (3.17)–(3.19), we conclude that, for each \( t \in [0, T] \),

\[
|\Phi(t)|^2 + \int_0^t |D_2\Phi(s)|^2 ds \\
\leq \int_0^t \left[ 4(1 + \alpha^2)^2 + 2\sqrt{\Pi(Z)} + 4|Y^{(2)}(s)|^2 \right] |\Phi(s)|^2 ds \\
\leq \left[ 4(1 + \alpha^2)^2 + 2\sqrt{\Pi(Z)} + 4 \left( Y^{(2)} \right)_{C([0,T;H])}^2 \right] \int_0^t |\Phi(s)|^2 ds.
\]

Then the uniqueness follows from the Gronwall’s lemma. Therefore, the proof of the proposition is complete.

Now we are in the position to prove Theorem 3.1.

**Proof of Theorem 3.1.** As in [6], since \( \Pi(Z) < \infty \), then the process \( \{N([0, t], Z), t \geq 0\} \) has only finite jumps in each finite interval of \( \mathbb{R}_+ \), i.e. there exist \( 0 < \tau_1 < \tau_2 < \cdots < \tau_n < \cdots \), and for \( k = 1, 2, \ldots, \tau_k \in \{t \in D_0; \ p(t) \in Z\} \). For each \( n \in \mathbb{N} \), it is easy to check that \( \tau_n \) is an \( (\mathcal{F}_t)_{t \geq 0}\)-stopping time and \( \tau_n \to \infty \), as \( n \to \infty \). By Proposition 3.1, for any \( T \in (0, \tau_1) \), there exists a unique weak solution \( Y \in C([0, T; H]) \cap L^2([0, T; V]) \) on \([0, \tau_1)\). Set

\[
X^1(t) = \begin{cases} 
Y(t), & t \in [0, \tau_1), \\
Y(\tau_1 -) + \sigma(Y(\tau_1 -), p(\tau_1)), & t = \tau_1.
\end{cases}
\]


Hence \( \{X^1(t), t \in [0, \tau_1]\} \) uniquely solves (2.4) in the time interval \([0, \tau_1]\). Further define

\[
\begin{cases}
X^2_0 := X^1(\tau_1), \\
p(t) := p(s + \tau_1), \\
D_p := \{t \in \mathbb{R}_+; t + \tau_1 \in D_p\} \\
\hat{F}_t := \mathcal{F}_{\tau_1+t}.
\end{cases}
\]

Note that \( \tau_2 - \tau_1 \in \{t \in D_p; \tilde{p}(t) \in Z\} \). Then we might construct a process \( \{X^2(t), t \in [0, \tau_2 - \tau_1]\} \) by the same way as for \( \{X^1(t), t \in [0, \tau_1]\} \). Thus let

\[
X^3(t) = \begin{cases}
X^1(t), & t \in [0, \tau_1], \\
X^2(t - \tau_1), & t \in [\tau_1, \tau_2],
\end{cases}
\]

it is a unique solution of (2.4) in the time interval \([0, \tau_2]\). Then the existence of the unique global weak solution follows from the above successive procedure. Thus, we complete the proof of the theorem.

\( \square \)

4. Invariant Measure

Based on the existence and uniqueness of the weak solution \( X \in L^\infty(0, T; H) \cap L^2(0, T; V) \) of (2.3), the existence of an invariant measure will be proved for \( X \) in this section. As for the Markov property of the solution process \( X \), the readers may refer to [8]. Let us define the transition semigroup of \( X \), for \( t \geq 0 \),

\[
\mathcal{P}_t \varphi(x) = \mathbb{E} \varphi(X(t; x)), \quad \text{for } x \in H,
\]

where \( \varphi \in C^b(H) \) (the space of all continuous and bounded functions on \( H \)). We call \( \{\mathcal{P}_t\}_{t \geq 0} \) admitting an invariant measure \( \nu \) on \( H \), if for each \( t > 0 \),

\[
\int_H \mathcal{P}_t \varphi(x) \nu(dx) = \int_H \varphi(x) \nu(dx), \quad \varphi \in C^b(H).
\]

First, we claim that the transition semigroup \( \{\mathcal{P}_t\}_{t \geq 0} \) is Fellerian under assumptions (H1) and (H3). Indeed, by Lemma 7.1.5 on p. 125 of [2], it suffices to show that for each \( \varphi \in C^2_b(H) \) (the space of all twice Fréchet differential functions with bounded and continuous derivatives up to second order), there exists a \( C := C(t) > 0 \) such that, for each fixed \( t \), \( |\mathcal{P}_t \varphi(x) - \mathcal{P}_t \varphi(y)| \leq C|x - y| \). Clearly, by the proof of Theorem 3.1, \( \sup_{t\in[0,T]} |X(t)|^2 \) is bounded \( \mathbb{P} \)-a.s. under the assumption (H3). Hence from Itô rule and by the similar estimates as in (3.3)–(3.4) and (3.18), there exists a \( K > 0 \) such that, for \( x, y \in H \),

\[
\mathbb{E} |X(t; x) - X(t; y)|^2 + \mathbb{E} \int_0^t |X(s; x) - X(s; y)|^2 ds \leq |x - y|^2 + [4(1 + \alpha^2) + K] \int_0^t \mathbb{E} |X(s; x) - X(s; y)|^2 ds.
\]
Thus the Gronwall’s lemma shows that there exists a $C := C(t) > 0$ such that for each fixed $t \geq 0$,
\[ \mathbb{E}|X(t; x) - X(t; y)|^2 \leq C|x - y|^2, \]
and so
\[ |\mathcal{P}_t \varphi(x) - \mathcal{P}_t \varphi(y)|^2 \leq \|\varphi\|_{\infty} \mathbb{E}|X(t; x) - X(t; y)|^2 \leq C\|\varphi\|_{\infty}|x - y|^2. \]

Now, we state our main result of this section:

**Theorem 4.1.** Under the assumptions (H1)–(H3) in Sec. 2, there exists an invariant measure $\nu$ on $H$ for $\{\mathcal{P}_t\}_{t \geq 0}$. Moreover, there exists a constant $C > 0$, such that
\[
\begin{align*}
\int_H |x|^2 \nu(dx) &\leq C, \\
\int_H |x|^2 \nu(dx) &\leq C.
\end{align*}
\]

In order to prove Theorem 4.1, we quote a new version of the Gronwall’s inequality from [7].

**Lemma 4.1.** Suppose that $v(t)$ is a positive differentiable function, and for some continuous functions $f, h$ in $[\alpha, \beta]$ $(\alpha < \beta \in \mathbb{R})$ and every $p \geq 0$ $(p \neq 1)$,
\[
\frac{dv(t)}{dt} \leq f(t)v(t) + h(t)v^p(t), \quad t \in [\alpha, \beta),
\]
holds. Then
\[
v(t) \leq \exp \left( \int_{\alpha}^{t} f(s)ds \right) \left[ v^q(\alpha) + q \int_{\alpha}^{t} h(s) \exp \left( -q \int_{\alpha}^{s} f(\tau)d\tau \right) ds \right] ^{\frac{1}{q}},
\]
for all $t \in [\alpha, \beta_1)$, where $q = 1 - p$ and $\beta_1$ is chosen so that the above expression between $[\ldots]$ is positive in the subinterval $[\alpha, \beta_1)$.

Next we turn to the proof of Theorem 4.1.

**Proof of Theorem 4.1.** Apply Itô formula to $|X(t)|^2$, we obtain, for $t > 0$,
\[
\begin{align*}
|X(t)|^2 &= |x|^2 + 2 \int_{0}^{t} \langle -D_4 X(s) - D_2 X(s), X(s) \rangle - (B(X(s)), X(s)) ds \\
&\quad - 2\alpha \int_{0}^{t} \langle \mathcal{H}(D_3 X(s)), X(s) \rangle ds + \int_{0}^{t} \int_{Z} |\sigma(X(s), z)|^2 \Pi(dz) ds \\
&\quad + \int_{0}^{t} \int_{Z} \langle (\sigma(X(s), z))^2 + 2(\sigma(X(s), z), X(s)) \rangle \tilde{N}(dz, dt) \\
&= |x|^2 - 2 \int_{0}^{t} |X(s)|^2 ds + 2 \int_{0}^{t} |X(s)|^2 ds
\end{align*}
\]
By a similar proof as for (3.3) and the interpolation inequality (2.2), we have,

\[ -2\alpha \int_0^t \langle H(D_3X(s)), X(s) \rangle \, ds + \int_0^t \int_Z |\sigma(X(s), z)|^2 \Pi(dz) \, ds \]

\[ + \int_0^t \int_Z \left( |\sigma(X(s), z)|^2 + 2|\sigma(X(s), z), X(s)| \right) \tilde{N}(dz, ds). \]  

(4.4)

By a similar proof as for (3.3) and the interpolation inequality (2.2), we have,

\[ \left| \alpha \int_G \mathcal{H}(D_3X(\xi, s))d\xi \right| \leq \frac{1}{4}|D_2X(s)|^2 + \alpha^2|D_1X(s)|^2, \]

(4.5)

and

\[ 2(1 + \alpha^2)|DX(s)|^2 \leq 2(1 + \alpha^2)|X(s)||D_2X(s)| \]

\[ \leq \frac{1}{2}|X(s)|^2 + 2(1 + \alpha^2)|X(s)|^2. \]  

(4.6)

Recall that \( C^* = \int_Z |\sigma(0, z)|^2 \Pi(dz) \). Then for all \( t > 0 \),

\[ \mathbf{E}|X(t)|^2 \leq \mathbf{E}|x|^2 - \int_0^t \mathbf{E}|X(s)|^2 ds \]

\[ + \left[ 2(1 + \alpha^2)^2 + \ell^2 \right] \int_0^t \mathbf{E}|X(s)|^2 ds + C^* t \]

\[ \leq \mathbf{E}|x|^2 - \lambda_1^2 \int_0^t \mathbf{E}|X(s)|^2 ds + (\lambda_1^2 - \gamma) \int_0^t \mathbf{E}|X(s)|^2 ds + C^* t \]

\[ = \mathbf{E}|x|^2 - \gamma \int_0^t \mathbf{E}|X(s)|^2 ds + C^* t, \]  

(4.7)

where \( \gamma = \lambda_1^2 - (2(1 + \alpha^2)^2 + \ell^2) > 0 \) is defined in Sec. 2, and so

\[ \frac{d\mathbf{E}|X(t)|^2}{dt} \leq -\gamma \mathbf{E}|X(t)|^2 + C^*. \]  

(4.8)

Then by Lemma 4.1,

\[ \mathbf{E}|X(t)|^2 \leq \exp\{ -\gamma t \} \mathbf{E}|x|^2 + \frac{C^*}{\gamma}. \]  

(4.9)

Thus from (4.7) and (4.9), it follows that for each \( t > 0 \),

\[ \frac{1}{t} \int_0^t \mathbf{E}|X(s)|^2 ds \leq \frac{1}{t} \left[ \mathbf{E}|x|^2 + C^* t \right] \]

\[ + \left[ 2(1 + \alpha^2)^2 + \Pi(Z)\ell^2 \right] \left( 1 - e^{-\gamma t} \right) \left[ (1 - e^{-\gamma t}) \mathbf{E}|x|^2 C^* t \right] . \]

Hence for \( T \geq T_0 > 0 \) and \( \epsilon > 0 \) small enough, we have,

\[ \frac{1}{T} \int_0^T \mathbf{P} \left( |X(s)|^2 > R \right) ds \leq \frac{1}{R^2} \left( \frac{1}{T_0} \mathbf{E}|x|^2 \right) \]

\[ \quad + \frac{1}{R^2} \left[ 2(1 + \alpha^2)^2 + \Pi(Z)\ell^2 \right] \left( \frac{1}{T_0} \mathbf{E}|x|^2 + C^* \right) \]

\[ \leq \epsilon, \quad \text{if } R \text{ is large enough.} \]  

(4.10)
This implies that the family
\[
\left\{ \frac{1}{T} \int_0^T P_t(x, \cdot) dt; \ T \geq T_0 \right\}
\]
is uniformly tight for each fixed \( x \in H \), since the embedding \( V \subset H \) is compact (see e.g. [12]). By the Krylov–Bogoliubov Theorem (see e.g. [2]) and the Feller property of the semigroup \( \{P_t\}_{t \geq 0} \), there exists an invariant measure for \( \{P_t\}_{t \geq 0} \). For any \( \varepsilon > 0 \) and \( y \in \mathbb{R}_+ \), define
\[
Z_\varepsilon(y) := \frac{y}{1 + \varepsilon y}.
\]
Then we have,
\[
Z_\varepsilon'(y) = \frac{1}{(1 + \varepsilon y)^2} \quad \text{and} \quad Z_\varepsilon''(y) = \frac{-2 \varepsilon}{(1 + \varepsilon y)^3}.
\]
Let \( \mathcal{X}(t) = |X(t; x)|^2 \). It follows from the Itô formula that,
\[
Z_\varepsilon(\mathcal{X}(t)) = Z_\varepsilon(\mathcal{X}(0)) - 2 \int_0^t Z_\varepsilon'(\mathcal{X}(s)) \langle D_4 X(s), X(s) \rangle ds
- 2 \int_0^t Z_\varepsilon'(\mathcal{X}(s)) \langle D_2 X(s), X(s) \rangle ds
- 2 \int_0^t Z_\varepsilon'(\mathcal{X}(s)) \langle B(X(s)), X(s) \rangle ds
- \alpha \int_0^t Z_\varepsilon'(\mathcal{X}(s)) \langle \mathcal{H}(D_3 X(s)), X(s) \rangle ds
+ \int_0^t \int_Z Z_\varepsilon'(\mathcal{X}(s)) |\sigma(X_s, z)|^2 \Pi(dz) ds
+ \int_0^t \int_Z Z_\varepsilon(\mathcal{X}(s-)) |\sigma(X(s-), z)|^2 + 2|\sigma(X(s-), z), X(s)|^2 + 2(\sigma(X(s-), z), X(s)) \Pi(dz) ds
- Z_\varepsilon(\mathcal{X}(s-)) \tilde{N}(dz, ds)
\]
\[
+ \int_0^t \int_Z Z_\varepsilon(\mathcal{X}(s) + |\sigma(X(s), z)|^2 + 2(\sigma(X(s), z), X(s)))
- Z_\varepsilon(\mathcal{X}(s))
- Z_\varepsilon'(\mathcal{X}(s)) \left( |\sigma(X(s), z)|^2 + 2(\sigma(X(s), z), X(s)) \right) \Pi(dz) ds.
\] (4.11)
If set
\[
G(x, z) = |\sigma(x, z)|^2 + 2(\sigma(x, z), x), \quad \text{for} \ x \in H,
\]
then the last term on the right-hand side of (4.11) might be rewritten as
\[
\int_0^1 (1 - \theta) d\theta \int_0^t \int_Z Z_\varepsilon''(\mathcal{X}(s) + \theta G(X(s), z)) |G(X(s), z)|^2 \Pi(dz) ds.
\]
Note that for any \( x \in H, z \in Z \) and \( \theta \in (0, 1), \)
\[
Z''_\varepsilon \left( |x|^2 + \theta G(x, z) \right) |G(x, z)|^2 \\
= \frac{-2\varepsilon |G(x, z)|^2}{(1 + \varepsilon |x|^2 + \varepsilon \theta |\sigma(x, z)|^2 + 2\varepsilon \theta (\sigma(x, z), x))^3} \\
= \frac{-2\varepsilon |G(x, z)|^2}{(1 + \varepsilon (1 - \theta) |x|^2 + \varepsilon \theta (|x|^2 + |\sigma(x, z)|^2 + 2(\sigma(x, z), x))^3} \\
\leq 0.
\]

Take expectations on both sides of (4.11), then
\[
E |X(t; x)|^2 \\
\leq \int_0^t \frac{|X(s; x)|^2}{(1 + \varepsilon |X(s; x)|^2)^2} ds \\
\leq \frac{|x|^2}{1 + \varepsilon |x|^2} + \left[ 2(1 + \alpha^2)^2 + \ell^2 \right] \int_0^t \frac{|X(s; x)|^2}{(1 + \varepsilon |X(s; x)|^2)^2} ds \\
+ \int_0^t \frac{C^*}{(1 + \varepsilon |X(s; x)|^2)^2} ds.
\]  
(4.12)

Since, for each \( t \in \mathbb{R}_+ \),
\[
\int_H E \psi(X(t; x)) \nu(dx) = \int_H \psi(x) \nu(dx), \quad \text{for all } \psi \in C_b(H),
\]
then
\[
\int_H \frac{|x|^2}{(1 + \varepsilon |x|^2)^2} \nu(dx) \leq \frac{C^*}{\gamma}.
\]  
(4.13)

Applying the Fatou lemma for \( \varepsilon \to 0 \) to conclude that
\[
\int_H |x|^2 \nu(dx) \leq \frac{C^*}{\gamma}.
\]  
(4.14)

Further, by (4.12), (4.13) and the Fatou’s lemma,
\[
\int_H |x|^2 \nu(dx) \leq \left[ 2(1 + \alpha^2)^2 + \ell^2 \right] \frac{C^*}{\gamma} + C^*.
\]  
(4.15)

Hence (4.3) follows from the estimates (4.14) and (4.15). This proves the theorem.

\[ \square \]

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