Exponential change of measure applied to term structures of interest rates and exchange rates

Lijun Bo
Department of Mathematics, Xidian University, Xi’an 710071, China

ABSTRACT

In this paper, we study the term structures of interest rates and foreign exchange rates through establishing a state-price deflator. The state-price deflator considered here can be viewed as an extension to the potential representation of the state-price density in [Rogers, L.C.G., 1997]. The potential approach to the term structure of interest rates and foreign exchange rates. Mathematical Finance 7(2), 157–164. We identify a risk-neutral probability measure from the state-price deflator by a technique of exponential change of measure for Markov processes proposed by [Palmowski, Z., Rolski, T., 2002]. A technique for exponential change of measure for Markov processes. Bernoulli 8(6), 767–785] and present examples of several classes of diffusion processes (jump–diffusions and diffusions with regime-switching) to illustrate the proposed theory. A comparison between the exponential change of measure and the Esscher transform for identifying risk-neutral measures is also presented. Finally, we consider the exchange rate dynamics by virtue of the ratio of the current state-price deflators between two economies and then discuss the pricing of currency options.

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1. Introduction

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space, where \(\mathbb{P}\) refers to a real world probability measure. In Rogers (1997) and Duffie (2001), a state-price density process \(\xi = \{\xi_t : 0 \leq t \leq T\}\) was established directly to value bonds under a stochastic interest rate environment. The state-price deflator is a positive super-martingale\(^1\) such that the deflated gain process associated with any admissible trading strategy is a \(\mathbb{P}\)-martingale. Then for any contingent claim \(Y\) payable at maturity \(T\), the time-\(t\) value of this claim can be given by

\[
P^\mathbb{F}(t, T) := \xi_t^{-1} \mathbb{E}_t [\xi_T Y], \quad 0 \leq t \leq T, \tag{1}
\]

where \(\mathbb{E}[\cdot] := \mathbb{E}[\cdot | \mathcal{F}_t]\) denotes the real world conditional expectation given the information \(\mathcal{F}_t\) available at time \(t\). If the claim \(Y = 1\) payable at maturity \(T\), the value \(P(t, T) := P^\mathbb{F}(t, T)\) matches the time-\(t\) price of a zero-coupon default-free bond that promises to pay one unit of account at maturity \(T\). When the contingent claim has the representation \(Y = \mathbb{1}_{\tau < T}\), where \(\tau\) represents the default time of a zero-coupon defaultable bond, the value \(P^\mathbb{F}(t, T)\) corresponds to the time-\(t\) price of this bond with zero recovery (see, e.g., Duffie, 2001 and Lando, 2004).

In addition, Rogers (1997) showed that the foreign exchange rate can also be identified by the ratio of respective state-price deflators over two countries (see also, Baksht et al., 2008). Let \(Q^\mathbb{P}_i\) represent the time-\(t\) currency-\(i\) price of the currency-\(j\). Here we assume that \(i\) is the domestic economy. Then for any \(h > 0\),

\[
\frac{Q^\mathbb{P}_{i+h}}{Q^\mathbb{P}_i} = \xi_{i+h}^{\mathbb{P}} / \xi_i^{\mathbb{P}}, \tag{2}
\]

where \(\xi_i^{\mathbb{P}}\) and \(\xi_{i+h}^{\mathbb{P}}\) denote the state-price deflators of the domestic and foreign economies, respectively. The term structure of exchange rate \(Q^\mathbb{P}_i\) will be discussed in Section 7.

On the other hand, the state-price deflator can be linked to a risk-neutral probability measure \(\mathbb{P}\) in terms of the stochastic discounted factor (see Rogers, 1997 and Duffie, 2001)

\[
\frac{d\mathbb{P}}{d\mathbb{P}} \bigg|_{\mathcal{F}_t} = \xi_t \exp \left( \int_0^t r_u du \right), \quad 0 \leq t \leq T, \tag{3}
\]

where \(r_t\) is the nonnegative stochastic spot interest rate at time \(t\). Thus the time-\(t\) price \(P^\mathbb{P}(t, T)\) determined by (1) becomes

\[
P^\mathbb{P}(t, T) = \mathbb{E}_t \left[ \exp \left( - \int_t^T r_u du \right) Y \right], \quad 0 \leq t \leq T. \tag{4}
\]

This is a well-known arbitrage-free pricing formula for the \(\mathcal{F}_T\)-adapted nonnegative random variable \(Y\).

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\(^1\) The interest rate should be assumed to be nonnegative.

E-mail address: bolijunnk@yahoo.com.cn.
To evaluate various derivatives associated with interest rates and exchange rates (such as zero-coupon bonds, currency options and embedded options in life insurance), the term structures of interest rates and exchange rates have become an important research area in the modern financial theory. In particular, in the valuation of life insurance products with embedded options (such as interest rate guarantees, surrender options and bonus policies), it is also important to incorporate a stochastically changing term structure of interest rate (see, e.g., Nielsen and Sandmann, 1995, Grosen and Jørgensen, 2000, Jørgensen, 2001 and references therein). Some well-known term structure models of interest rates include the Vasicek model in Vasicek (1977), the CIR model in Cox et al. (1985), the Ho–Lee model in Ho and Lee (1986), the Hull–White model in Hull and White (1990) and so on (the readers can refer to Chapter 7 of Duffie (2001) for more term structure models). For the term structures of exchange rates, the pioneering works are due to Biger and Hull (1983) and Garman and Kohlhagen (1983), in which the dynamics of exchange rate is modeled as a geometric Brownian motion. Afterward it was extended to jump–diffusion models by Jorion (1988), Heston (1993) and Johnson and Schneeweiß (1994), and to stochastic volatility models by Mellino and Turnbull (1990). Most recently, Siu et al. (2008) proposed a regime-switching stochastic volatility exchange rate model to price relevant currency options.

In this paper, we consider the nonnegative random spot interest rate \( r = \{ r_t : 0 \leq t \leq T \} \) to depend on some underlying Markov process \( X = \{ X_t : 0 \leq t \leq T \} \) which takes values in a Borel space \( S \). It means that the random interest rate \( r_t \) can be rewritten as

\[
 r_t = r(X_t), \quad 0 \leq t \leq T,
\]

where \( r(\cdot) \) is a nonnegative measurable function on the Borel space \( S \). Herein, we establish the state-price deflator \( \xi \) by the following form

\[
 \xi_t = \frac{f(X_t)}{f(X_0)} \exp \left\{ - \int_0^t \left( r(X_u) + \frac{A f(X_u)}{f(X_0)} \right) \, du \right\},
\]

where \( A \) is the (extended) generator of \( X \) under the real world probability \( \mathbb{P} \) and \( f \) is taken to be a (nonnegative) good function (whose definition can be found in the second paragraph of page 768 in Palmowski and Rolski, 2002). In Section 2, we shall state the motivation to why we specify the state-price deflator as the above representation (5). A main reason is that the state-price deflator (5) indeed extends the potential representation of the state-price density in Rogers (1997) and hence we have a larger class of spot interest rate functions when we consider the term structures and pricing of interest rate and exchange rate derivatives.

Using the link between (3) and (5), we can identify a risk-neutral probability measure \( \mathbb{Q} \) by

\[
 \frac{d\mathbb{Q}}{d\mathbb{P}} \bigg|_{r_t} = \Phi_f(t) := \exp \left( \int_0^t r(X_u) \, du \right) \xi_t^{f,r},
\]

where the likelihood ratio has the form

\[
 \Phi_f(t) = \frac{f(X_t)}{f(X_0)} \exp \left( - \int_0^t \frac{Af(X_u)}{f(X_0)} \, du \right), \quad 0 \leq t \leq T.
\]

This likelihood ratio corresponds to the density process of the Palmowski and Rolski (2002)'s exponential change of measure for the underlying Markov process \( X \). In particular, the (extended) generator of \( X \) can be re-identified under the risk-neutral probability measure \( \mathbb{Q} \). This result is useful in the risk-neutral valuation of interest rate and exchange rate derivatives.

In Section 3, we compare the forward measure related to the current state-price deflator (5) to the forward measure identified by (2.9) in Rogers (1997). Sections 4 and 5 are devoted to presenting examples of two classes of diffusion processes for illustrating the risk-neutral valuation. One class is the multi-dimensional (jump) diffusion process as the underlying Markov process, and the other refers to the regime-switching jump–diffusion as the underlying Markov process. We identify the respective risk-neutral measures \( \mathbb{Q} \) by choosing different good functions defined in Palmowski and Rolski (2002). In Section 6, we give a comparison between the exponential change of measure and the Esscher transform adopted in Elliott et al. (2005), Elliott et al. (2007) and Siu et al. (2008) when the underlying Markov process is assumed to follow a geometric jump–diffusion with regime-switching. The exchange rate dynamics in terms of (2) is considered in Section 7. Applying the exponential change of measure, we also prove that there exists a class of dynamics for the exchange rate under the domestic risk-neutral probability measure, which includes the Garman–Kohlhagen exchange rate model (see Garman and Kohlhagen, 1983). Further, in some simple setting, the risk-neutral price representation of currency options can be reduced to the Garman–Kohlhagen formula (see Garman and Kohlhagen, 1983 and Shreve, 2008 for the derivation of the formula).

2. Introducing a general state-price kernel

In this section, we introduce a general state-price deflator (5) and provide the motivation for this approach. We begin with the potential representation of the state-price density (2.1) in Rogers (1997). Rogers (1997) defined a specified spot random interest rate (see (2.7) in Rogers, 1997)

\[
 r_t = r(X_t) = \frac{(\alpha_d - A) f(X_t)}{f(X_0)},
\]

where \( I_\alpha \) denotes the identical operator, \( \alpha > 0 \) and \( f : \mathbb{S} \rightarrow (0, \infty) \) is defined on \( D(A) \), the domain of definition of the generator \( A \). To guarantee the positivity of the interest rate, Rogers had assumed that \( (\alpha, f) \) satisfies

\[
 g(x) := (\alpha_d - A) f(x) \geq 0, \quad \forall x \in \mathbb{S}.
\]

Define the resolvent of the generator \( A \) by \( R_\alpha = (\alpha_d - A)^{-1} \) with \( \alpha > 0 \). Then the interest rate (8) can be rewritten as

\[
 r_t = r(X_t) = \frac{g(X_t)}{R_\alpha g(X_0)},
\]

The corresponding state-price density (see (2.1) in Rogers, 1997) given by

\[
 \xi_t = e^{-\alpha t} \frac{R_\alpha g(X_t)}{R_\alpha g(X_0)},
\]

is hence a positive \( \mathbb{P} \)-super-martingale. (This then also implies that the above \( \xi \) is a \( \mathbb{P} \)-local martingale.) However, we find that the positive super-martingale defined by (10) can be extended to the more general \( \mathbb{P} \)-super-martingale (5). To illustrate this point, substitute Rogers’ interest rate formula (8) into the current state-price deflator (5) to give

\[
 \xi_t = e^{-\alpha t} \frac{f(X_t)}{f(X_0)} f(X_0) = e^{-\alpha t} R_\alpha g(X_t) R_\alpha g(X_0).
\]

This form is consistent with the state-price density (2.1) in Rogers (1997). Accordingly, we can say that the potential representation of the state-price density in Rogers (1997) is a special case of the current state-price deflator (5).
Secondly, we can identify a risk-neutral likelihood ratio with respect to the real world probability measure $\mathbb{P}$ by a technique of exponential change of measure proposed by Palmowski and Rolski (2002). Indeed, Palmowski and Rolski (2002) showed that under mild conditions, the underlying Markov process $X$ is also Markov under the risk-neutral measure $\mathbb{P}^f$ defined by (6), whose $\mathbb{P}^f$-generator has the form

$$\tilde{A} h(x) = \left[ A(\tilde{f}h) - h A\tilde{f} \right](x). \quad (12)$$

In particular, the likelihood ratio $\Phi_f(t)$ admits some explicit forms when the (nonnegative) good function $f(\cdot)$ satisfies some additional conditions. For example, if the good function $f(\cdot)$ satisfies

$$A f = \alpha f, \quad \alpha > 0,$$

this results in

$$\frac{\text{d}\mathbb{P}^f}{\text{d}\mathbb{P}} \bigg|_{\mathcal{F}_t} = e^{-\alpha f(X_t)}/f(X_0). \quad (13)$$

For the likelihood ratio (13), we can produce instances of the classic exponential change of measure. The special choice of the good function $f(\cdot)$ yields Girsanov’s theorem (see, e.g., Rogers and Williams, 2000).

**Example 1.** The underlying Markov process $X$ is a $d$-dimensional Brownian motion $X_t = (W_{t1}, \ldots, W_{td})$ with $d \in \mathbb{N}$. Taking a good function to be

$$f(x_1, \ldots, x_d) = \exp \left( \sigma \sum_{i=1}^d x_i \right),$$

where $\sigma > 0$ is a positive constant. Then $A f = \frac{\text{d}^2 f}{\text{d}x^2}$ and the likelihood ratio is then

$$\Phi_f(t) = \exp \left( \sigma \sum_{i=1}^d W_{ti} - \frac{\sigma^2}{2} t \right).$$

Let $\tilde{W}_t := W_{ti} - \alpha t$ for $i = 1, \ldots, d$. Then $\tilde{W}_t = (\tilde{W}_{t1}, \ldots, \tilde{W}_{td})$ is a $d$-dimensional $\mathbb{P}^f$-Brownian motion by using (12).

**Example 2.** For the discontinuous underlying Markov process, a candidate is $X_t = N_t$, Poisson process with intensity $\lambda > 0$. For a constant $\sigma > 0$, taking a good function $f(n) = \exp(\sigma n)$ on $[0] \cup \mathbb{N}$, we have

$$A f(n) = \lambda f(n+1) - f(n) = \lambda (e^{\sigma} - 1)f(n).$$

This results in the likelihood ratio

$$\Phi_f(t) = \exp \left( \sigma N_t - \lambda (e^{\sigma} - 1) t \right).$$

The generator (12) further shows that the Poisson process $N_t$ admits the $\mathbb{P}^f$-intensity $\lambda e^\sigma$.

By the exponential change of measure (6), the time-$t$ price $P(t, T)$ of the zero-coupon default-free bond given by (4) can be re-identified under the risk-neutral probability measure $\mathbb{P}^f$,

$$P(t, T) = \mathbb{E}^f \left[ \exp \left( - \int_t^T r(X_u) \text{d}u \right) \right], \quad 0 \leq t \leq T. \quad (14)$$

Moreover, if we consider a zero-coupon defaultable bond with zero recovery under a reduced-form credit risk model (see, e.g., Duffie, 2001 and Lando, 2004), the time-$t$ price follows

$$P^d(t, T) = \mathbb{E}^f \left[ \exp \left( - \int_t^T r(X_u) \text{d}u \right) \right], \quad 0 \leq t \leq T, \quad (15)$$

where $(T - r)\psi(\cdot)$ is the function of the default arrival intensity. In particular, when the (adjusted) interest rate $(\tilde{T}, r)$ is specified by (9), the valuation of the bonds is reduced to the case of Rogers (1997).

### 3. Forward measures

In this section, we discuss forward measures in view of the discounted price of a zero-coupon bond and compare the forward measure associated to the current state-price deflator (5) with the one in Rogers (1997).

As in Shreve (2008), we can define the $T$-forward measure $\mathbb{P}_T^f$ by

$$\frac{\text{d}\mathbb{P}_T^f}{\text{d}\mathbb{P}} \bigg|_{\mathcal{F}_t} = \exp \left( - \int_0^t r(X_u) \text{d}u \right) \frac{P(t, T)}{P(0, T)}. \quad (16)$$

Using (7) and (14), it follows that

$$\frac{\text{d}\mathbb{P}_T^f}{\text{d}\mathbb{P}} \bigg|_{\mathcal{F}_t} = \Phi_f^{-1}(t) \frac{\mathbb{E}_0[\xi_T]}{\mathbb{E}_0[\xi_T]}, \quad (17)$$

where $\xi_T$ is the state-price density (5). Note that $\Phi^{-1}_f(t) = \tilde{T}$ with

$$\tilde{T}_f^{-1}(t) := \frac{f(X_0)}{f(X_t) - \tilde{f}(X_0)} \exp \left( - \int_0^t f(X_u) \tilde{w}^{-1}(X_u) \text{d}u \right).$$

If $f^{-1}(\cdot)$ is also a good function, then $\mathbb{E}_0[\xi_T]$ is a $\mathbb{P}^f_T$-martingale by virtue of Lemma 4.1 in Palmowski and Rolski (2002).

Next we shall give a concrete form of the generator of the underlying Markov process $X$ under the forward measure $\mathbb{P}_T^f$. The following result also gives a main difference from (2.13) in Rogers (1997).

**Proposition 3.1.** Let $g^{x,f}(x) = (r + Af(t))x$, where $x \in S$. Assume that $f(\cdot)$ is a good function. Then the generator of $X$ under the forward measure $\mathbb{P}_T^f$ is given by

$$A_{x}^f \psi(t, x) = \left[ A(\psi) - \varphi A\psi + \frac{\partial \varphi}{\partial t} \right](t, x),$$

$$\left(t, x \right) \in [0, T] \times S. \quad (18)$$

Here the function $\psi(\cdot, \cdot)$ satisfies the equation

$$\left( \frac{\partial}{\partial t} + A \right) \psi(t, x) - g^{x,f}(x) \psi(t, x) = 0, \quad (19)$$

$$\psi(T, x) = \frac{f(x)}{f(X_0)P(0, T)}, \quad$$

where $x_0 \in S$ is the initial value of the underlying Markov process $X$.

**Remark 3.1.** We compare (19) with (2.13) in Rogers (1997), and find that Rogers’ forward-generator (2.13) is a special case of (19) with the assumption that $g^{x,f}(\cdot) \equiv 0$. This is consistent with the previous discussions of (8)–(11) in Section 2.

**Proof.** From (5) and (17), it follows that there exists a measurable function $\psi(\cdot, \cdot)$ on $[0, T] \times S$ such that

$$\frac{\text{d}\mathbb{P}_T^f}{\text{d}\mathbb{P}} \bigg|_{\mathcal{F}_t} = \exp \left( - \int_0^t g^{x,f}(X_u) \text{d}u \right) \psi(t, X_t), \quad (20)$$

where $\mathbb{P}$ is the real world probability measure defined in Section 1. Apply Itô’s formula to the above likelihood ratio to conclude that

$$\text{d} \left[ \exp \left( - \int_0^t g^{x,f}(X_u) \text{d}u \right) \psi(t, X_t) \right] = \exp \left( - \int_0^t g^{x,f}(X_u) \text{d}u \right) d\psi(t, X_t) - \psi(t, X_t) g^{x,f}(X_t).$$
become a instead of:

\[ \psi(\exp(t \theta X_t^d)) = \psi(t X_t^d) \exp \left( -\int_0^t \alpha dW^d_s \right) \]

where \( \alpha = (\alpha_1, \ldots, \alpha_d)^T \) and \( X_t^d = (X_t^1, \ldots, X_t^d)^T \) are jointly Wiener processes. Let \( \theta 

Theorem 4.1. Continuous diffusion as an underlying Markov process

Let \( W_t = (W_t^1, \ldots, W_t^m)^T \) be an \( m \)-dimensional Brownian motion on the real world probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Suppose the underlying Markov process \( X \) satisfies the following \( d \)-dimensional stochastic differential equation (SDE):

\[ dX_t = (b(X_t) dt + \sigma(X_t) dW_t) \]

where \( b(x) = (b_1(x), \ldots, b_m(x))^T \) and \( \sigma(x) = \sigma_1(x) \sigma_2(x) \ldots \sigma_m(x) \) have the Lipschitz conditions. This implies that \( \psi \) is an arbitrary smooth function with compact support on \([0, T] \times S\).

Using Girsanov’s theorem, we have that under the risk-neutral measure \( \mathbb{P}^\mathbb{Q} \), the process \( \tilde{W}_t^i = W_t^i - \frac{1}{2} \sum_{k=1}^m \sigma_{ik}(X_t) \sigma_k(X_t) dW_t^k - \int_0^t \theta_s^i \frac{\partial}{\partial t} \psi \frac{\partial}{\partial x} \psi \ dt \)

are mutually independent Brownian motions. In addition, the state-price deflator given by (5) follows:

\[ \tilde{X}_t^i = \mathbb{E}^\mathbb{Q} \left[ 1 + \sum_{i=1}^m \int_0^t \theta_s^i \frac{\partial}{\partial t} \psi \frac{\partial}{\partial x} \psi \ dt \right] \]

By (12), we have that under the risk-neutral measure \( \mathbb{P}^\mathbb{Q} \), the generator of \( X \) is given by

\[ \tilde{A} h(x) = \mathbb{E}^\mathbb{Q} \left[ A(fh) - hA \psi \right] \]

where \( h \in C^2(\mathbb{R}^d) \). This implies that under \( \mathbb{P}^\mathbb{Q} \), the underlying Markov process \( X \) satisfies the SDE:

\[ dX_t = (b(X_t) dt + \sigma(X_t) dW_t) \]

Thus we complete the proof of the proposition.

4. Examples: diffusion process as an underlying Markov process

In this section, we consider the case of a multi-dimensional diffusion process as the underlying Markov process \( X \).
4.2. Jump–diffusion as an underlying Markov process

For parsimony, we here consider one-dimensional jump–diffusion process as the underlying Markov process $X$.

Let $W^T$ be a one-dimensional Brownian motion and $N(d\rho, dt)$ be a Poisson random measure with the compensator $\nu(d\rho)dt$ supported by the common real world probability space. Suppose the Brownian motion and Poisson random measure are mutually independent under $\mathbb{P}$. Consider the following jump–diffusion as the underlying Markov process $X$,

$$dX_t^k = \left( b_k(X_t) + \sum_{i=1}^{d} c_{ik}(X_t) a_i \right) dt + \sum_{i=1}^{m} \sigma_{ik}(X_t) d\tilde{W}^i_t,$$

$$X_0 = x_0 \in \mathbb{R}^d,$$

where the $m$-dimensional $\mathbb{F}$-Brownian motion $\tilde{W}_t = (\tilde{W}^1_t, \ldots, \tilde{W}^m_t)$ is defined by (25). If $a = (a_1, \ldots, a_d)$ is a vector function with respect to $x$, the corresponding likelihood ratio can be found in Section 5.3 of Palmowski and Rolski (2002).

We next turn to Rogers (1997)’s interest rate model (8). Then the corresponding Rogers’ interest rate function $r(\cdot)$ is given by

$$r(x) = \alpha - \sum_{i=1}^{d} b_i(x) a_i - \frac{1}{2} \sum_{i=1}^{d} c_{ik}(x) a_i a_k,$$

(28)

if the above interest rate function $r(\cdot)$ is nonnegative. When the multi-dimensional diffusion $X$ is affine and such that the function $r(\cdot)$ is nonnegative, $r(\cdot)$ is also affine. This characteristic is useful in the valuation of interest rate derivatives (see Rogers, 1997 and Duffie, 2001).

Next we are concerned with the time-$t$ price of a zero-coupon bond $P(t, T)$ given by (14). Similarly we can also consider the price $P^b(t, T)$ of a defaultable zero-coupon bond. Define the risk-neutral price function by

$$F(t, x; b, \sigma, r) := \mathbb{F} \left[ \exp \left( - \int_0^T r(X_u) du \right) | X_t = x \right],$$

(29)

where $\mathbb{F}[\cdot | X_t = x]$ is the conditional expectation with respect to the risk-neutral probability measure $\mathbb{F}$ identified by (24). Using Feynman–Kac formula, we conclude that this price function satisfies

$$\left( \frac{\partial}{\partial t} + A^{b, \sigma} \right) F(t, x; b, \sigma, r) = r(x) F(t, x; b, \sigma, r),$$

(30)

where $A^{b, \sigma}$ denotes the $\mathbb{F}$-generator of $X$ given by (26). Hence the risk-neutral price of the zero-coupon bond is $P(t, T) = F(t, X_t; b, \sigma, r)$, where $b(x) = (b_1(x), \ldots, b_d(x))^T$ and $\sigma(x) = (\sigma_{ij}(x))_{1 \leq i \leq d, 1 \leq j \leq m}$.

5. Examples: regime-switching diffusion as an underlying Markov process

5.1. Brownian motion with regime-switching

This subsection concentrates on a Markov modulated Brownian motion (BM) as the underlying Markov process $X$.

Let $W = \{ W_t : 0 \leq t \leq T \}$ be a Brownian motion and $\varepsilon = \{ \varepsilon(t) : 0 \leq t \leq T \}$ be a continuous-time finite-states Markov chain, which is independent of $W$ under $\mathbb{P}$. The BM with regime-switching is given by

$$dY_t = \mu_{\varepsilon(t)} dt + \sigma_{\varepsilon(t)} dW_t,$$

(37)
The underlying Markov process is taken to be $X_t = (Y_t, \varepsilon(t))$. Here we assume the Markov chain $\varepsilon$ has only two states, say $[0, 1]$ and whose $\mathbb{P}$-generator is given by
\[
\begin{bmatrix}
-\lambda_0 & \lambda_0 \\
\lambda_1 & -\lambda_1
\end{bmatrix},
\]
where $\lambda_0$ and $\lambda_1$ are two positive constants. To identify the state-price density (5), we first compute the $\mathbb{P}$-generator of the joint Markov process $X_t$,
\[
Ah_t(y) = h_t'(y)\mu_t + \frac{1}{2}h_t''(y)\sigma_t^2 + \lambda_i[h_{1-i}(y) - h_i(y)],
\]
i = 0, 1, (38)
where $h_0(\cdot)$, $h_1(\cdot) \in C^2(\mathbb{R})$. Similarly to the case of Markov additive processes in Section 5.4 of Palmowski and Rolski (2002), we express (nonnegative) good functions in the form of
\[
f_i(y) = \exp(ay)\beta_i(a), \quad i = 0, 1,
\]
where $a \in \mathbb{R}$ and for a fixed constant $a$, $\beta_i(a) > 0$ which will be specified later. Then we have
\[
A\Psi_i(y) = f_i(y) \left[ \psi_i(a) + \lambda_i \left( \frac{\beta_{1-i}(a)}{\beta_i(a)} - 1 \right) \right], \quad i = 0, 1,
\]
where $\psi_i(a) := a\mu_i + \frac{\sigma_i^2}{2}$, for $i = 0, 1$.

For a fixed constant $a$, let $\gamma(a)$ be the real Perron–Frobenius eigenvalue (which is unique and has the maximal real part by Theorem 1.6.5 in Asmussen, 2003) and $(\hat{\beta}_1(a), \hat{\beta}_2(a))$ be the positive corresponding right eigenvector of the matrix
\[
\begin{bmatrix}
\psi_1(a) - \lambda_0 & \lambda_0 \\
\lambda_1 & \psi_2(a) - \lambda_1
\end{bmatrix}.
\]
That is
\[
\begin{bmatrix}
\psi_1(a) - \lambda_0 & \lambda_0 \\
\lambda_1 & \psi_2(a) - \lambda_1
\end{bmatrix} = \gamma(a) \begin{bmatrix}
\hat{\beta}_1(a) \\
\hat{\beta}_2(a)
\end{bmatrix}.
\]
Here for a fixed constant $a$, we take the unspecified positive numbers $(\beta_1(a), \beta_2(a)) = (\hat{\beta}_1(a), \hat{\beta}_2(a))$ in (39). As a consequence, the likelihood ratio (6) becomes
\[
d\mathbb{P}_{\gamma_t} = \Phi_f(t) = \exp[a(Y_t - Y_0) - \gamma(a)t] \frac{\beta_{i+1}(a)}{\beta_i(a)}. (41)
\]
Further, the form (12) yields that the $\mathbb{P}$-generator of the joint Markov process $X_t$ is
\[
\mathbb{A}\hat{h}_t(x) = \left[ A(h_t) - h_tA_t \right](x) = \left( \mu_t + a^2f_i(y) \frac{f_i(y)}{f_t(y)} \right) h_i(y) + \frac{\sigma_i^2}{2}h_i''(y)
+ \lambda_i f_{1-i}(y) [h_{1-i}(y) - h_i(y)] = (\mu_t + a^2 h_i''(y) + \frac{\sigma^2_i}{2}h_i''(y)
+ \lambda_i \frac{\beta_{1-i}(a)}{\beta_i(a)} [h_{1-i}(y) - h_i(y)], (42)
\]
where $i = 0, 1$. This implies that under $\mathbb{P}$, the joint Markov process $X_t = (Y_t, \varepsilon(t))$ can be represented as
\[
dY_t = \left( \mu_{x(t)} + a^2 \right) dt + \sigma_{x(t)}dW_t, (43)
\]
where $W_t := W_t - a \int_0^t \sigma_{x(t)}(u) du$ is a $\mathbb{P}$-Brownian motion and the Markov chain $\varepsilon$ has the $\mathbb{P}$-generator
\[
\begin{bmatrix}
-\lambda_0 & \lambda_0 \\
\lambda_1 & -\lambda_1
\end{bmatrix}.
\]
As for Rogers (1997)'s interest rate model (8), the corresponding interest rate function $r(\cdot)$ is given by
\[
r_t(x) = \alpha - \psi_1(a) - \lambda_i \left( \frac{\beta_{1-i}(a)}{\beta_i(a)} - 1 \right), \quad i = 0, 1, (45)
\]
if the above interest rate functions $r_0(\cdot)$, $r_1(\cdot)$ are nonnegative.

Next we discuss the time-t price $P(t, T)$ given by (14) when the underlying Markov process $X_t$ is the joint Markov process $(Y_t, \varepsilon_t)$. As in Section 4, define the risk-neutral price function by
\[
F_t(t, \gamma_t; \mu, \sigma, \lambda, r) := \mathbb{E}\left[ \exp\left( -\int_0^T r(u)(X_u) du \right) \right] Y_t = \gamma(t) = \delta(t).
\]
where $\delta(\cdot)$ denotes the $\mathbb{P}$-generator given by (42). By (43), $P(t, T) = F_{x(t)}(t, \mu, \sigma, \lambda, r)$.

5.2. Geometric Brownian motion with regime-switching

When the underlying Markov process $X_t$ is the following geometric Brownian motion (GBM) with regime-switching
\[
dY_t = \mu_{x(t)} dt + \sigma_{x(t)} dW_t, \quad Y_0 = y_0 > 0,
\]
the time-t price $P(t, T)$ given by (14) can be identified as the following equation
\[
\left( \frac{\partial}{\partial t} + \mathbb{A}^{\mu-\frac{1}{2}\sigma^2, \sigma, \lambda} \right) G_t(t, x; \mu, \sigma, \lambda, r)
= r_t \left( \exp(x) \right) G_t(t, x; \mu, \sigma, \lambda, r)
\]
where $\mathbb{A}^{\mu-\frac{1}{2}\sigma^2, \sigma, \lambda}$ is the $\mathbb{P}$-generator given by (42) but with $\mu - \frac{1}{2}\sigma^2$ instead of $\mu$. Accordingly the time-t price $P(t, T) = G_{x(t)}(t, \mu, \sigma, \lambda, r)$.

Under Rogers' potential approach, Grazioso and Rogers have considered the option pricing problems with Markov modulated dynamics in Grazioso and Rogers (2006) (see also Jobert and Rogers, 2006).

5.3. Jump–diffusion with regime-switching

This subsection discusses a geometric jump–diffusion with regime-switching expressed as
\[
\frac{dY_t}{Y_t} = \mu_{x(t)} dt + \sigma_{x(t)} dW_t + \int_{\Lambda} \gamma_{x(t)}(\rho) \tilde{N}(d\rho, dt),
\]
where $\gamma(\cdot) > -1$ (i = 0, 1) and $\tilde{N}(d\rho, dt) := N(d\rho, dt) - \gamma_{x(t)}(\rho)\mu_t d\rho, dt$ is a compensated Poisson random measure with the Markov modulated compensator $\gamma_{x(t)}(d\rho, dt)$. Here we assume random shocks $(W, N)$ are conditionally independent given knowledge of the Markov chain under $\mathbb{P}$.
To identify a risk-neutral probability measure, we first consider the following log jump–diffusion process with regime-switching
\[ dL_t = \left( \mu_{\alpha(t)} - \frac{1}{2} \sigma_{\alpha(t)}^2 \right) dt + \sigma_{\alpha(t)} dW_t - \int_A \gamma_{\alpha(t)}(\rho) v_{\alpha(t)}(d\rho) dt \]
\[ + \int_A \log(\gamma_{\alpha(t)}(\rho) + 1) N(d\rho, dt). \tag{51} \]
As in Section 5.1, we propose good functions
\[ f_\epsilon(z) = \exp(az) \beta_1(z), \quad z \in \mathbb{R}, \]
where \( a \in \mathbb{R} \) and \( (\beta_1(z), \beta_2(z)) \) is defined the same as in Section 5.1. Then
\[ A f_\epsilon(z) = f_\epsilon(z) \left[ \psi_i(a) + \lambda_i \left( \beta_1(z) - 1 \right) \right], \tag{52} \]
where for \( i = 0, 1 \),
\[ \psi_i(a) := a \left( \mu_i - \frac{1}{2} \sigma_i^2 \right) + \frac{a^2}{2} \sigma_i^2 \]
\[ + \int_A \left[ \gamma_i(\rho) + 1 \right]^i - 1 - a \gamma_i(\rho) \right] v_i(d\rho), \]
where the constant \( a \) is taken to guarantee the finiteness of \( \psi_i(a) \).

Let \( \gamma(a) \) be defined the same as in Section 5.1. Then the likelihood ratio (6) has the form
\[ \frac{d\mathbb{P}^\gamma}{d\mathbb{P}} \bigg|_{\mathfrak{F}_t} = \Psi_t(t) = \exp \left[ a(L_t - L_0) - \gamma(a)t \frac{\beta_1(t)}{\beta_0(t)} \right]. \tag{53} \]

Under the risk-neutral measure \( \mathbb{P}^\gamma \), the generator of the joint Markov process \( (L_t, \epsilon(t)) \) is
\[ \mathcal{A}^\gamma h(z) = \left[ \mathcal{A}(fh) - h(A \epsilon(t)) \right](z) \]
\[ = \frac{1}{2} h''(z) \sigma^2 \]
\[ + \int_A \left[ \gamma(z) + 1 \right]^i - 1 - a \gamma_i(\rho) \right] v_i(d\rho), \]
where the constant \( a \) is taken to guarantee the finiteness of \( \psi_i(a) \).

At the moment, the time- \( t \)- price \( P(t, T) \) given by (14) can be identified as
\[ \frac{\partial}{\partial t} A^\mu, \sigma, \nu, \lambda \bigg|_{\mathfrak{F}} G_t(t, x; \mu, \sigma, v, \nu, \lambda, r) \]
\[ = r_i(\exp(x)) G_t(t, x; \mu, \sigma, v, \nu, \lambda, r) \tag{55} \]
where \( A^\mu, \sigma, \nu, \lambda \) is the \( \mathbb{P}^\gamma \)-generator of the log jump–diffusion (51).

Then \( P(t, T) = G_{\epsilon(t)}(t, L_t; \mu, \sigma, v, \nu, \lambda, r) \).

6. Esscher transforms

The method of Esscher transforms is a time-honored technique in actuarial science. It has been shown by Gerber and Shiu (1994, 1996), Bühlmann et al. (1998) and others that it is also an efficient method to price financial derivatives. In particular, Yao (2001) and Dijkstra and Yao (2002) studied the pricing of interest rate and exchange rate claims by Esscher transform and moment generating function approaches respectively. Therein the state-price density is modeled as an exponential function of the underlying state variables.

In this section, we use the Esscher transform adopted by Elliott et al. (2005, 2007) and Siu et al. (2008) to identify a risk-neutral probability measure when the underlying Markov process is expressed as the Markov modulated geometric jump–diffusion (50). Moreover, we compare the difference between the Esscher transform and the exponential change of measure implemented in Section 5.3.

Recall the Markov modulated geometric jump–diffusion (50) in Section 5.3. We decompose its log jump–diffusion process (51) by \( L_t = C_t + J_t \), \( 0 \leq t \leq T \), where the continuous and jump parts are respectively defined by
\[ C_t := L_0 + \int_0^t \left( \mu_{\sigma(u)} - \frac{1}{2} \sigma^2_{\sigma(u)} \right) du + \int_0^t \sigma_{\sigma(u)} dW_u \]
\[ - \int_0^t \int_A \gamma_{\sigma(u)}(\rho) v_{\sigma(u)}(d\rho) du, \]
\[ J_t := \int_0^t \int_A \log(\gamma_{\sigma(u)}(\rho) + 1) N(d\rho, du). \]

In terms of (2.5) in Bo et al. (2010), we specify \( \Gamma_t^{c, J} \) by the Esscher transform as follows:
\[ \frac{d\mathbb{Q}}{d\mathbb{P}} \bigg|_{\mathfrak{F}_t} = \exp \left[ \int_0^t \theta_u^c \delta^c_u dC_u + \int_0^t \theta_u^J \delta^J_u dI_u \right]. \tag{56} \]

where \( \theta_u^c := \theta_u^{c,i} \) with \( i \in \{ c, J \} \) are respective Markov modulated market prices of risk for the continuous and jump parts. Because of the denominator on the R.H.S. of (56), the Esscher transform adopted in Elliott et al. (2005, 2007) and Siu et al. (2008) can be carried out only when the whole sample path of the underlying Markov chain is given. This point can be seen in the following calculations.

Note that
\[ \mathbb{E} \left[ \exp \left( \int_0^t \theta_u^c \delta^c_u dC_u \right) \right] = \exp \left[ \int_0^t \theta_u^c \left( \mu_{\sigma(u)} - \frac{1}{2} \sigma^2_{\sigma(u)} \right) \right]. \]
\[ \times \exp \left( \int_0^t \theta_u^{J,i} \sigma^2_{\sigma(u)} dI_u \right). \]
and
\[\mathbb{E}\left[ \exp \left( \int_0^t \theta_u \, dW_u \right) \bigg| \mathcal{F}_t \right] = \exp \left( \int_0^t \int_A \left( \gamma_u(\rho) \right) \nu_{(i\omega)}(d\rho) \, du \right).\]

As a consequence
\[R_{t}^{ij} = \exp \left( \int_0^t \int_A \theta_u \sigma_u^{ij} \, dW_u - \frac{1}{2} \int_0^t \left( \theta_u \right)^2 \sigma_u^{ij} \, du \right) \times \exp \left( \int_0^t \int_A \theta_u \log(\gamma_u(\rho)) \, dW_u \right) \times \exp \left( - \int_0^t \int_A \left( \gamma_u(\rho) \right) \nu_{(i\omega)}(d\rho) \, du \right).\]

By virtue of Girsanov’s theorem, we have under the probability measure \(Q\) identified by the Esscher transform (56),
\[W_{t}^{i} := W_{t} - \int_{0}^{t} \theta_{u} \sigma_{u}^{ij} \, dW_{u} \text{ is a Brownian motion, and}
N^{i}(d\rho, dt) := N(d\rho, dt) - \left( \gamma_{(i\omega)}(\rho) + 1 \right)^{d} \nu_{(i\omega)}(d\rho, dt) \text{ is a compensated Poisson random measure. (58)}\]

Comparing \(W^{i}\) (resp. \(N^{i}\)) defined by (58) with \(\bar{W}\) (resp. \(\bar{N}\)) defined by (54) in Section 5.3, we find that the Brownian motions \((W^{i}\) and \(W)\) and Poisson random measures \((N^{i}\) and \(N)\) under the respective change of the real world probability measure admit the consistent forms, respectively. However, the generator of the Markov chain under \(\mathbb{P}\) changes in contrast to the one under the real world probability measure \(\mathbb{P}\) and while the generator remains unchanged from the real world measure \(\mathbb{P}\) to the probability measure \(Q\) identified by the Esscher transform (56).

7. Exchange rates

Rogers (1997) proved that the foreign exchange rate can be expressed as the ratio of respective state-price densities over two economies (see Appendix of Rogers, 1997 or Bakshi et al., 2008).

As in Section 1, the \(Q^{\mu}\) is the time-\(t\) currency-i price of the currency-j and \(i\) is assumed to be the domestic economy. In this section, we express the respective state-price deflators over two economies \(i\) and \(j\) as
\[\xi_{t}^{k} := \frac{f_{k}(X_{t})}{f_{k}(X_{0})} \exp \left\{ - \int_{0}^{t} \left( r_{k}(X_{u}) + \frac{\sigma_{k}^{2}(X_{u})}{2} \right) \, du \right\}, \]
\[k \in \{i, j\},\]
where \(r_{k}(\cdot)\) and \(\xi_{t}^{k}\) denote the respective spot interest rate functions of the domestic and foreign economies. Here \(f_{i}(\cdot)\) and \(f_{j}(\cdot)\) are taken to be (nonnegative) good functions. Then by Rogers’ exchange rate representation (2) in Section 1, we have
\[\frac{Q_{t}^{\mu}}{Q_{0}^{\mu}} = \frac{\xi_{t}^{i}}{\xi_{t}^{j}} = \frac{f_{i}(X_{t})}{f_{j}(X_{t})} \frac{f_{j}(X_{0})}{f_{i}(X_{0})} \exp \left\{ - \int_{0}^{t} \left( d_{i} r_{i}(X_{u}) \right) \right\},\]
where \(d_{i} r_{i}(\cdot) := (r_{i} - r_{j})(\cdot)\) denotes the interest rate differential function.

Next we consider the dynamics of \(Q^{\mu} = \{Q_{t}^{\mu} : 0 \leq t \leq T\}\) when the underlying Markov process \(X\) is the jump–diffusion (31) and the (nonnegative) good functions are taken to be
\[f_{k}(x) = \exp(a_{k} x), \quad a_{k} \in \mathbb{R}, \quad k \in \{i, j\}.\]

Then we have the following.

Lemma 7.1. The exchange rate \(Q^{\mu}\) has the following representation under the real world probability measure \(\mathbb{P}\),
\[Q_{t}^{\mu} = Q_{0}^{\mu} \exp \left\{ (a_{i} - a_{j}) \left( \int_{0}^{t} \sigma(X_{u}) dW_{u} \right) \right\} \times \exp \left\{ - \int_{0}^{t} \int_A \gamma_{(i\omega)}(\rho) \nu_{(i\omega)}(d\rho) \, du \right\} \right\},\]
where \(\bar{W}_{t}^{i} := W_{t} - a_{j} \int_{0}^{t} \sigma(X_{u}) dW_{u} \) and \(\bar{N}_{t}^{i}(d\rho, dt) := N(d\rho, dt) - e^{r_{(i\omega)}(X_{t})} v(d\rho, dt) \text{ are the domestic risk-neutral Brownian motion and compensated Poisson random measure, respectively.}\]

Similarly, we can further extend Lemma 7.1 to the case of the underlying Markov process \(X\) being the regime-switching jump–diffusion (50).\(^4\)

Proof. The representation (61) under the real world probability measure \(\mathbb{P}\) can be derived by virtue of (60), the forms of good functions \(f_{k}(\cdot)\) and the jump–diffusion (31). Note that the expression (61) is indeed equivalent to the following form of SDE under \(\mathbb{P}\),
\[\frac{dQ_{t}^{\mu}}{Q_{t}^{\mu}} = (r_{i} - r_{j})(X_{t}) dt + a_{j} \sigma(X_{t}) dW_{t} \]
\[+ \int_A \left( e^{r_{(i\omega)}(X_{t})} - 1 \right) \bar{N}_{t}(d\rho, dt) \]
\[+ \int_A \left( e^{r_{(j\omega)}(X_{t})} - e^{r_{(i\omega)}(X_{t})} \right) \bar{N}_{t}(d\rho, dt) + \frac{1}{2} \sigma_{(i\omega)}^{2}(X_{t}) dt,\]
where \(a_{i} := a_{i} - a_{j} \neq 0\). Then the domestic risk-neutral dynamics (62) follows from (35) in Section 4.2. □

In what follows, we consider a call on unit of foreign currency whose time-\(t\) claim in the domestic currency is \((Q^{\mu} - K)^{+}\) with the dynamics of the exchange rate proved by Lemma 7.1. The value of the call at time \(t\) is given by

\(^4\) We thank the referee for pointing this out to us.
\[ V^d(t) = \frac{1}{\xi_t} \mathbb{E}_t \left[ \xi_T \left( q_T^d - K \right)^+ \right] \]
\[ = \mathbb{E}_t \left[ \exp \left( - \int_t^T r_t \, du \right) \left( q_u^d - K \right)^+ \right], \quad 0 \leq t \leq T. \] (64)

As discussed in the previous sections, the value \( V^d(t) \) can be identified as the Feynman–Kac formula. In particular, in some simple setting, we can obtain a closed-form pricing formula. For example, if the domestic and foreign interest rates are positive constants, \( \sigma(x) \equiv \sigma > 0 \) and \( \gamma(\rho, x) \equiv \rho \) (this implies that the exchange rate has not jumps and hence the underlying Markov process \( X \) is a GBM), the time-\( t \) price \( V^d(t) \) is
\[ V^d(t) = e^{-\gamma(T-t)} Q_t^d N(d_1(t)) - e^{-\gamma(T-t)} K N(d_2(t)), \] (65)
where \( N(\cdot) \) denotes the cumulative standard normal distribution function and
\[ d_{1,2}(t) := \frac{\log \left( \frac{q_t^d}{q_t^f} \right) + (r_t - \frac{1}{2} \sigma^2 t) (T-t)}{\sigma \sqrt{T-t}}. \] (66)

The price representation (65) is called the Garman–Kohlhagen formula (see, e.g., Garman and Kohlhagen, 1983, Dumas et al., 1995 and Shreve, 2008).

Finally, we discuss the price \( V^d(t) \) when jumps exist in the exchange rate dynamics (62). For convenience, we here suppose the domestic and foreign interest rates are positive constants, \( \sigma(x) \equiv \sigma > 0 \) and \( \gamma(\rho, x) = \rho \in \mathbb{R}/\{0\} \). Then the dynamics of exchange rate \( Q_t^d \) under the domestic risk-neutral measure \( \mathbb{P}^d \) can be written as
\[ \frac{dQ_t^d}{Q_t^d} = (r_t - \gamma) dt + \sigma dW_t^d + \int_{\mathbb{R}/\{0\}} (e^{\omega} - 1) N^d(d\omega, dt), \] (67)
where \( W_t^d = W_t - a_t \) and \( N^d(d\omega, dt) = N(d\omega, dt) - e^{\omega} N(d\omega, dt) \). Define the risk-neutral price process by
\[ V(t, q) := \mathbb{E}^{d} \left[ e^{-\gamma(T-t)} \left( q_T^d - K \right)^+ \bigg| q_t = q \right], \quad q > 0. \]

Then \( V(t, q) \) satisfies the following integro-differential equation (IDE):
\[
\begin{aligned}
\frac{\partial r_t V(t, q)}{\partial t} + \frac{\partial V(t, q)}{\partial q} \gamma q_t - \frac{1}{2} \frac{\partial^2 V(t, q)}{\partial q^2} \sigma^2 q_t \\
+ \int_{\mathbb{R}/\{0\}} e^{\omega} \left[ V(t, q e^{\omega}) - V(t, q) \right] \, d\omega(q, dq) \\
- q(e^{\omega} - 1) \frac{\partial V(t, q)}{\partial q} \, dq,
\end{aligned}
\]
\[ V(t, q) = (q - K)^+, \quad q > 0. \] (68)

Hence the time-\( t \) value of the call currency option is \( V^d(t) = V(t, Q_t^d) \). Now the IDE (68) with appropriate boundary conditions can be solved numerically (see, e.g., Duffie, 2001).

8. Conclusions

This paper investigated the term structures of interest rates and foreign exchange rates by establishing a general state-price deflator. The state-price deflator considered here is an extension to the potential representation of the state-price density in Rogers (1997). We discussed the risk-neutral valuation problem associated with interest rate and exchange rate derivatives by a technique of exponential change of measure for Markov processes proposed by Palmowski and Rolski (2002). When the underlying Markov process of the state-price deflator was expressed as a geometric jump–diffusion model with regime-switching, a comparison between the exponential change of measure and the Esscher transform for identifying risk-neutral measures was given. In light of the state-price deflator considered herein, we established the term structures of exchange rates, which include the Garman–Kohlhagen exchange rate model.

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References