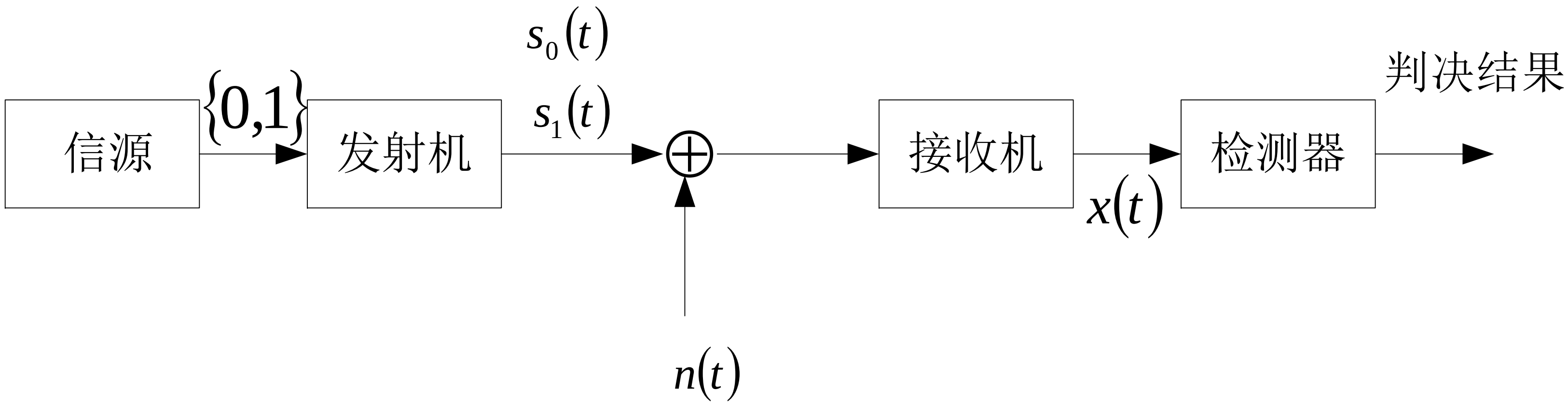
第四章 波形信号检测

- 二元信号波形检测模型
- 匹配滤波器
- 正交级数展开
- 波形信号检测

二元信号波形检测模型



信源输出 **发射信号**

0 $s_0(t),\quad nT \le t \le (n+1)T$

1 $s_1(t),\quad nT \le t \le (n+1)T$

信号在信道中传输，受到加性噪声的干扰，可描述为：

$$H_0:\quad x(t) = s_0(t) + n(t),\quad nT + t_0 \le t \le (n+1)T + t_0$$

$$H_1:\quad x(t) = s_1(t) + n(t),\quad nT + t_0 \le t \le (n+1)T + t_0$$

波形信号检测方法

波形信号

$x(t)$

随机信号正交级数展开

$l(x(t)) \underset{<}{\overset{>}{}} \gamma_c$

匹配滤波器

离散信号

$\{x_1, x_2, \ldots, x_N\}$

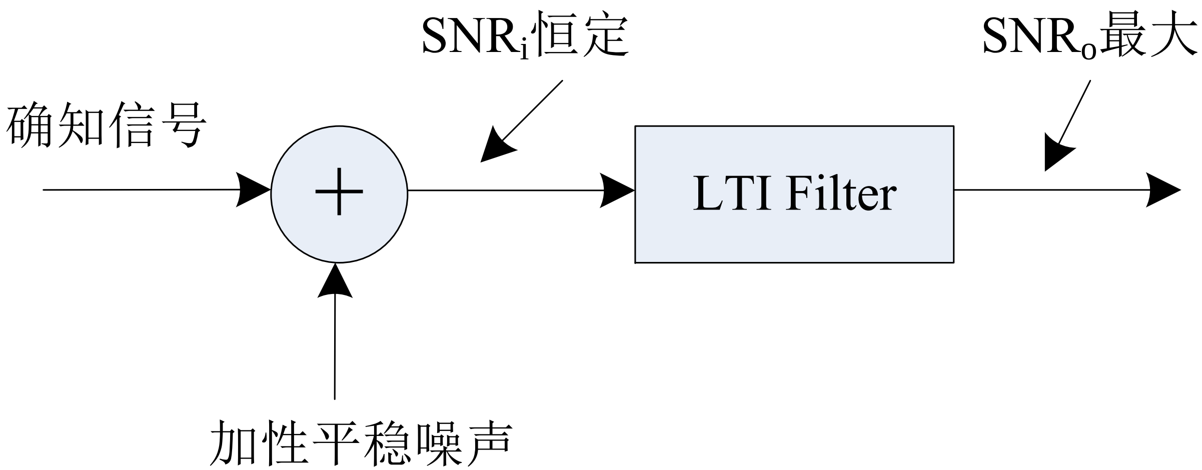
离散信号统计检测

$l(x_1^N) \underset{<}{\overset{>}{}} \gamma_d$

匹配滤波器（MF）

- 定义

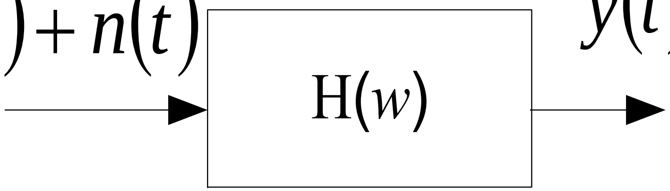
----若线性时不变（LTI）滤波器输入的信号是确知信号，噪声是加性平稳噪声，则在输入功率信噪比一定的条件下，使输出功率信噪比最大的滤波器，即为与输入信号匹配的最佳滤波器，称为匹配滤波器。



系统模型

- 假设线性时不变滤波器的冲激相应为**h(t)**，系统函数为***H*(*w*)**，滤波器的输入信号为**x(t)=s(t)+n(t)**，滤波器的输出信号为**y(t)=s$_o$(t)+n$_o$(t)** 。

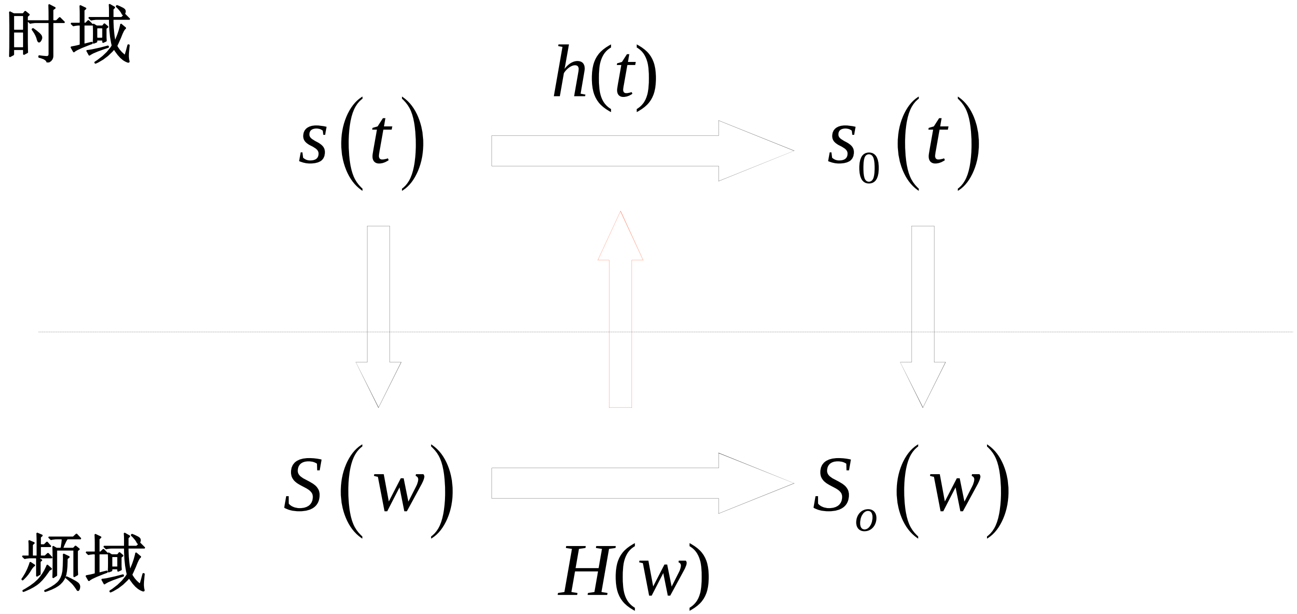
$$x(t)=s(t)+n(t) \qquad y(t)=s_o(t)+n_o(t)$$



$$S(\omega)=\int_{-\infty}^{\infty}s(t)e^{-j\omega t}dt \qquad E_s=\int_{-\infty}^{\infty}s^2(t)dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|S^2(\omega)\right|d\omega$$

- 设计目标：输出功率信噪比最大。

设计思路



$$S(w) = FT\{s(t)\} = \int_{-\infty}^{\infty} s(t) e^{-jwt} dt$$

$$s(t) = IFT\{S(w)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(w) e^{jwt} dw$$

Formulation

- 目标：输出信号功率信噪比定义为输出信号$s_o(t)$的峰值功率与噪声$n_o(t)$的平均功率之比。

$$SNR_O \overset{def}{=} \frac{s_o(t)\text{的功率}}{n_o(t)\text{的平均功率}}$$

- 已知条件：

$$s_o(t) = s(t) * h(t) \qquad S_o(\omega) = S(\omega)H(\omega)$$

$$E_S = \int_{-\infty}^{\infty} s^2(t)\,dt < \infty \qquad s_o(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_o(\omega) e^{j\omega t} d\omega$$

$$P_{n_o}(\omega) = |H(\omega)|^2 P_n(\omega) \qquad = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)H(\omega) e^{j\omega t} d\omega$$

Solution

- 假设输出信号在t_0出现峰值，则输出信号的峰值功率为

$$\left|s_o(t_0)\right|^2=\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}S(\omega)H(\omega)e^{j\omega t_0}d\omega\right|^2$$

- 输出噪声的平均功率为

$$E\left[n_o^2(t)\right]=\frac{1}{2\pi}\int_{-\infty}^{\infty}P_{n_o}(\omega)d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|H(\omega)\right|^2P_n(\omega)d\omega$$

Solution

$$SNR_O \overset{def}{=} \frac{s_o(t)\text{的功率}}{n_o(t)\text{的平均功率}}$$

$$= \frac{\left|\frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)S(\omega)e^{j\omega t_0}d\omega\right|^2}{\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|H(\omega)\right|^2 P_n(\omega)d\omega}$$

Cauchy-Schwarz不等式

$$\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}F^*(t)Q(t)dt\right|^2 \le \frac{1}{2\pi}\int_{-\infty}^{\infty}F^*(t)F(t)dt\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty}Q^*(t)Q(t)dt$$

等号成立的条件为

$$Q(t)=\alpha F(t)$$

$$F^*(\omega)=\frac{S(\omega)e^{j\omega t_0}}{\sqrt{P_n(\omega)}} \qquad Q(\omega)=H(\omega)\sqrt{P_n(\omega)}$$

Alternative form:

$$\left\|\langle a,b\rangle\right\| \le \|a\|\cdot\|b\|$$

Solution

变量替换

$$F^*(\omega)=\frac{S(\omega)e^{j\omega t_0}}{\sqrt{P_n(\omega)}} \qquad Q(\omega)=H(\omega)\sqrt{P_n(\omega)}$$

$$\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}H(\omega)S(\omega)e^{j\omega t_0}d\omega\right|^2=\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}H(\omega)\sqrt{P_n(\omega)}\frac{S(\omega)e^{j\omega t_0}}{\sqrt{P_n(\omega)}}d\omega\right|^2$$

$$\leq\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|H(\omega)\sqrt{P_n(\omega)}\right|^2 d\omega\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{S(\omega)e^{j\omega t_0}}{\sqrt{P_n(\omega)}}\right|^2 d\omega$$

Solution

$$SNR_O \overset{def}{=} \frac{s_o(t)\text{的功率}}{n_o(t)\text{的平均功率}} = \frac{\left|\frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)S(\omega)e^{j\omega t_0}d\omega\right|^2}{\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|H(\omega)\right|^2 P_n(\omega)d\omega}$$

$$\leq \frac{\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|H(\omega)\sqrt{P_n(\omega)}\right|^2 d\omega \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{S(\omega)e^{j\omega t_0}}{\sqrt{P_n(\omega)}}\right|^2 d\omega}{\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|H(\omega)\sqrt{P_n(\omega)}\right|^2 d\omega \cdot}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\left|S(\omega)\right|^2}{P_n(\omega)}d\omega$$

Solution

由于

$$F^*(\omega)=\frac{S(\omega)e^{j\omega t_0}}{\sqrt{P_n(\omega)}} \qquad Q(\omega)=H(\omega)\sqrt{P_n(\omega)}$$

所以，当 $Q(\omega)=\alpha F(\omega)$ 时，即

$$H(\omega)\sqrt{P_n(\omega)}=\alpha\frac{S^*(\omega)e^{-j\omega t_0}}{\sqrt{P_n(\omega)}}$$

$$H(\omega)=\alpha\frac{S^*(\omega)e^{-j\omega t_0}}{P_n(\omega)}$$

Solution

信道噪声为高斯白噪声的情况

$$P_n(\omega)=\frac{N_0}{2}$$

$$H(\omega)=2\alpha\frac{S^*(\omega)e^{-j\omega t_0}}{N_0}=kS^*(\omega)e^{-j\omega t_0}\quad h(t)=ks(t_0-t)$$

由**Parseval**定理

$$SNR_O=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\left|S(\omega)\right|^2}{N_0}d\omega=\frac{2E_s}{N_0}$$

Parseval定理

- **Parseval定理**

$$E_S=\int_{-\infty}^{\infty}\left|s(t)\right|^2dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|S(w)\right|^2dw$$

- **Proof:**

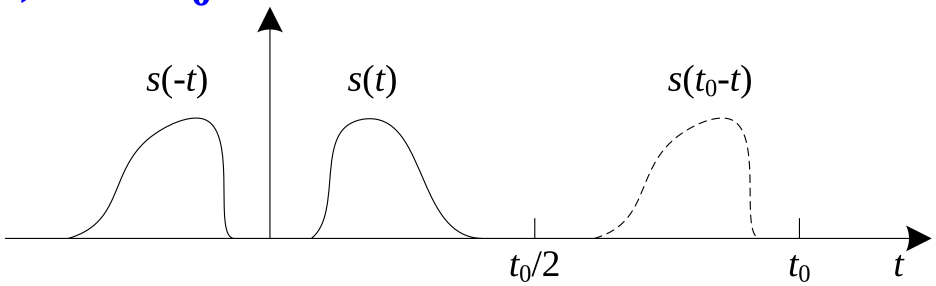
$$E_S=\int_{-\infty}^{\infty}\left|s(t)\right|^2dt=\int_{-\infty}^{\infty}s^*(t)\frac{1}{2\pi}\int_{-\infty}^{\infty}S(w)e^{jwt}dwdt$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}S(w)\int_{-\infty}^{\infty}s^*(t)e^{jwt}dtdw=\frac{1}{2\pi}\int_{-\infty}^{\infty}S(w)S^*(w)dw$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|S(w)\right|^2dw$$

MF的性质

- **h(t)与s(t)对于$t_0/2$呈对偶关系**



- **h(t)必须是物理可实现的，有**

$$h(t)=\begin{cases} s(t_0-t), & t\geq 0 \\ 0, & t<0 \end{cases}$$

- 为了确保输入信号**s(t)**的全部都能对输出信号有贡献，**t_0**应满足 $s(t)=0, t>t_0$

  t_0至少选择在**s(t)**的末尾

MF的性质

- t_0=T

$$h(t)=s(t_0-t),0\le t\le t_0$$

$$s_0(t)=\int_0^{t_0}h(\tau)s(t-\tau)d\tau$$

$$=\int_0^{t_0}s(t_0-\tau)s(t-\tau)d\tau$$

$$=\int_0^{t_0}s^2(u)du$$

- 既然s (t)的持续时间为(0,T)，选择t_0=T使得s_0 (t)最大，从而输出SNR最大。选取t_0>T不会改善输出SNR，只会延迟做出判决的时间

MF的性质

- 匹配滤波器的输出信号功率信噪比

$$SNR_O = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\left|S(\omega)\right|^2}{N_0}d\omega = \frac{2E_s}{N_0}$$

MF的性质

- 对于振幅和时延参量不同的信号，匹配滤波器具有适应性

$$H(\omega)=kS^*(\omega)e^{-j\omega t_0}$$

$$s_1(t)=As(t-\tau) \qquad S_1(\omega)=AS(\omega)e^{-j\omega\tau}$$

$$H_1(\omega)=kS_1^*(\omega)e^{-j\omega t_1}=AkS^*(\omega)e^{j\omega\tau}e^{-j\omega t_1}$$

$$=Ak\frac{H(w)}{e^{-j\omega t_0}k}e^{j\omega\tau}e^{-j\omega t_1}=AH(w)e^{-j\omega[t_1-(\tau+t_0)]}$$

$$=AH(w) \quad \text{if} \quad t_1=\tau+t_0$$

MF的性质

- 对于频移信号，匹配滤波器不具有适应性。

$$H(\omega)=kS^*(\omega)e^{-j\omega t_0}$$

$$s_1(t)=s(t)e^{jvt} \qquad S_1(\omega)=S(\omega+v)$$

$$H_1(\omega)=kS_1^*(\omega)e^{-j\omega t_1}=kS^*(\omega+v)e^{-j\omega t_1}$$

$v\neq 0$ 时，$H_1(\omega)$的频率特性与**H(w)**的频率特性不同。

MF的性质

- 匹配滤波器与相关器的关系

对于平稳输入信号 $x(t)=s(t)+n(t)$，自相关器的输出为：

$$r_x(\tau)=\int_{-\infty}^{\infty}x(t)x(t+\tau)dt$$

$$=\int_{-\infty}^{\infty}(s(t)+n(t))(s(t+\tau)+n(t+\tau))dt$$

$$=r_s(\tau)+r_n(\tau)+r_{sn}(\tau)+r_{ns}(\tau)$$

MF的性质

对于平稳输入信号 $x_1(t)=s(t)+n(t)$和$x_2(t)=s_0(t)$，互相关器的输出为：

$$r_{x_1x_2}(\tau)=\int_{-\infty}^{\infty}x_1(t)x_2(t+\tau)dt$$

$$=\int_{-\infty}^{\infty}(s(t)+n(t))s_0(t+\tau)dt$$

$$=r_{ss_0}(\tau)+r_{ns_0}(\tau)$$

MF的性质

假设本地信号为$s(t)$，相关器的输入信号 $x_1(t)=s(t)+n(t) \quad 0\le t\le T$

相关器的输出信号为

$$y_c(t)=\int_0^t x(u)s(u)du$$

$t\ge T$

$$y_c(t\ge T)=\int_0^T x(u)s(u)du$$

匹配滤波器的输出信号为 $h(t)=s(T-t)$

$$y_f(t)=\int_0^t x(t-\tau)h(\tau)d\tau=\int_0^t x(t-\tau)s(T-\tau)d\tau$$

$$y_f(t=T)=\int_0^T x(T-\tau)s(T-\tau)d\tau=\int_0^T x(u)s(u)du$$

Relation to Optimal detection

Minimal Distance Criterion

$$\boldsymbol{r} = \boldsymbol{s}_m + \boldsymbol{w}, m = 1, \ldots, M$$

$$\begin{aligned}
\hat{\boldsymbol{s}} &= \arg \min_{\boldsymbol{s}_m, m=1,\ldots,M} \left\| \boldsymbol{r} - \boldsymbol{s}_m \right\|^2 \\
&= \arg \min_{\boldsymbol{s}_m, m=1,\ldots,M} \left[\left\| \boldsymbol{r} \right\|^2 - 2 \langle \boldsymbol{r}, \boldsymbol{s}_m \rangle + \left\| \boldsymbol{s}_m \right\|^2 \right] \\
&= \arg \max_{\boldsymbol{s}_m, m=1,\ldots,M} \left[2 \langle \boldsymbol{r}, \boldsymbol{s}_m \rangle - \left\| \boldsymbol{s}_m \right\|^2 \right] \\
&= \arg \max_{\boldsymbol{s}_m, m=1,\ldots,M} \left[2 \langle \boldsymbol{r}, \boldsymbol{s}_m \rangle - E_m \right]
\end{aligned}$$

Relation to Optimal detection

- **Equal energy assumption**

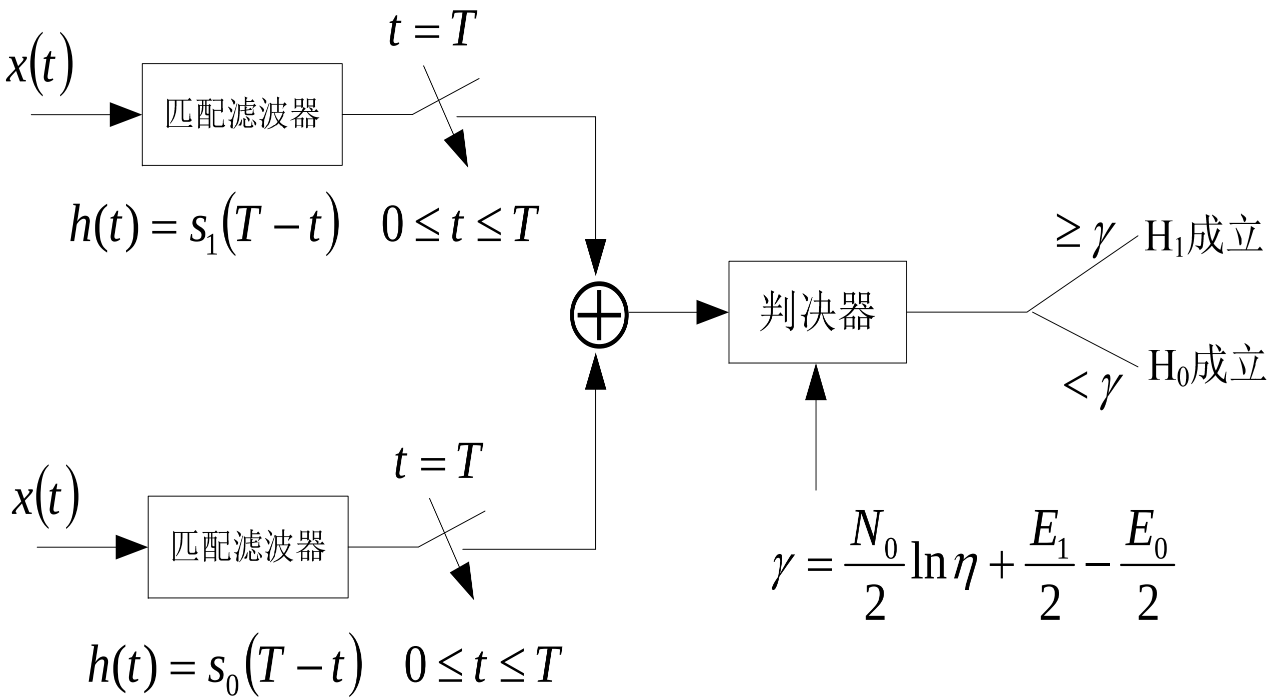
$$\hat{\boldsymbol{s}} = \arg \max_{\boldsymbol{s}_m, m=1,\dots,M} \left[2\langle \boldsymbol{r}, \boldsymbol{s}_m \rangle - E_m \right]$$

$$= \arg \max_{\boldsymbol{s}_m, m=1,\dots,M} \langle \boldsymbol{r}, \boldsymbol{s}_m \rangle$$

$$= \arg \max_{\boldsymbol{s}_m, m=1,\dots,M} \int_0^T r(t) s_m(t) dt$$

- **The minimal distance receiver can be implemented as a bank of matched filter**

检测系统结构



$$\gamma = \frac{N_0}{2}\ln\eta + \frac{E_1}{2} - \frac{E_0}{2}$$

波形信号检测方法

波形信号 → 离散信号

$x(t)$ → $\{x_1, x_2, \ldots, x_N\}$

随机信号正交级数展开

离散信号统计检测

$l(x(t)) \underset{<}{\overset{>}{}} \gamma_c$ → $l(x_1^N) \underset{<}{\overset{>}{}} \gamma_d$

匹配滤波器

Problem Formulation

- 考虑联合PDF $P\left(x_1^N \mid H_j\right)$

  如果x_k,k=1,…,N相互独立，则有

$$P\left(x_1^N \mid H_j\right)=\prod_{k=1}^{N} P\left(x_k \mid H_j\right)$$

- 目标：正交级数展开得到的系数相互独立

完备的正交函数集

若实函数集$\{f_k(t)\},k=1,2,\cdots$ 在**(0 ,T)**时间内满足

$$\int_0^T f_k(t)f_j(t)dt=\begin{cases}1, & j=k\\0, & j\neq k\end{cases}$$

且不存在归一化的非零函数**g(t)**，满足

$$\int_0^T f_k(t)g(t)dt=0$$

则称函数集 $\{f_k(t)\},k=1,2,\cdots$ 是完备正交函数集。

确知信号的正交级数展开

s(t)是定义在(0 ,T)时间内确知信号，且

$$E_s < \infty$$

该信号可用正交级数展开表示为：

$$s(t) = \lim_{N\to\infty} \sum_{k=1}^{N} s_k f_k(t)$$

$$s_k = \int_0^T f_k(t) s(t) dt$$

随机过程的正交级数展开

- 假设接收信号为

$$x(t)=s(t)+n(t)$$

其中**s(t)**是确知信号，**n(t)**是零均值、自相关函数为 $R_n(\tau)$ 的平稳随机过程，则接收信号也是平稳随机过程。

- 由于随机过程是由很多样本函数构成的集合，而每个样本函数是时间的函数，所以对给定的样本函数，可以进行正交级数展开

$$x(t)=\lim_{N\to\infty}\sum_{k=1}^{N}x_k f_k(t) \qquad x_k=\int_0^T f_k(t)x(t)dt$$

所有样本函数的展开系数，构成了一族随机变量。

随机过程的正交级数展开

- 目标：给出一种正交函数集的选择方法，以保证展开系数(高斯分布)之间是相互独立（互不相关）的随机变量。

假设随机过程为 $x(t)=s(t)+n(t)$ 正交函数集 $\{f_k(t)\}, k=1,2,\cdots$

$x(t)=s(t)+n(t)$ 的展开系数是随机变量，且

$$x_k=\int_0^T f_k(t)x(t)dt$$

$$E[x_k]=E\left[\int_0^T f_k(t)x(t)dt\right]=E\left[\int_0^T f_k(t)(s(t)+n(t))dt\right]$$

$$=s_k$$

随机过程的卡亨南-洛维展开

$$E[(x_k - s_k)(x_j - s_j)] = E\left[\left(\int_0^T f_k(t)x(t)dt - s_k\right)\left(\int_0^T f_j(t)x(t)dt - s_j\right)\right]$$

$$= E\left[\left(\int_0^T f_k(t)(s(t)+n(t))dt - s_k\right)\left(\int_0^T f_j(t)(s(t)+n(t))dt - s_j\right)\right]$$

$$= E\left[\left(\int_0^T f_k(t)n(t)dt\right)\left(\int_0^T f_j(t)n(t)dt\right)\right]$$

$$= E\left[\left(\int_0^T f_k(t)n(t)dt\right)\left(\int_0^T f_j(u)n(u)du\right)\right]$$

$$= \int_0^T f_k(t)\left[\left(\int_0^T E(n(t)n(u)) f_j(u)du\right)\right]dt$$

随机过程的卡亨南-洛维展开

$$E\left[\left(x_k - s_k\right)\left(x_j - s_j\right)\right] = \int_0^T f_k(t)\left[\left(\int_0^T E\left(n(t)n(u)\right) f_j(u)du\right)\right]dt$$

$$= \int_0^T f_k(t)\left[\left(\int_0^T R_n(t-u) f_j(u)du\right)\right]dt$$

为保证 $$E\left[\left(x_k - s_k\right)\left(x_j - s_j\right)\right] = \lambda_j \delta_{kj}$$

$$\int_0^T R_n(t-u) f_j(u)du = \lambda_j f_j(t)$$

白噪声情况下正交函数集的任意性

- 白噪声条件下

$$R_n(t-u)=\frac{N_0}{2}\delta(t-u)$$

$$E\left[\left(x_k-s_k\right)\left(x_j-s_j\right)\right]=\int_0^T f_k(t)\left[\left(\int_0^T R_n(t-u)f_j(u)du\right)\right]dt$$

$$=\int_0^T f_k(t)\left[\left(\int_0^T \frac{N_0}{2}\delta_n(t-u)f_j(u)du\right)\right]dt$$

$$=\frac{N_0}{2}\int_0^T f_k(t)f_j(t)dt=\frac{N_0}{2}\delta_{kj}$$

- 在白噪声条件下，可任意选取正交函数集，均可保证展开系数之间是不相关的。

一般二元信号的波形检测

- 信号模型

在一般二元信号的波形检测中，假设**H0**下和假设**H1**的接收信号分别为：

$$H_0: \quad x(t) = s_0(t) + n(t), \quad 0 \le t \le T$$

$$H_1: \quad x(t) = s_1(t) + n(t), \quad 0 \le t \le T$$

其中$s_0(t)$是能量为E_0的确知信号， $s_1(t)$是能量为E_1的确知信号
$n(t)$是均值为零，功率谱密度为$N_0/2$的高斯白噪声。

检测方法概述

- 首先，利用随机过程的正交级数展开，将随机过程用一组随机变量来表示；

- 然后，针对展开得到的随机变量，利用第一章的统计检测方法，构建贝叶斯检测表达式；

- 最后，利用展开系数与随机过程之间的表示关系，构建波形信号的检测表达式。

判决表达式

$$H_0: \quad x(t) = s_0(t) + n(t), \quad 0 \le t \le T$$

$$H_1: \quad x(t) = s_1(t) + n(t), \quad 0 \le t \le T$$

步骤1，选一组完备的正交函数集 $\{f_k(t), k = 1,2\cdots\}$
对接收信号进行正交级数展开，得到一组随机变量

$x_k, k = 1,2\cdots$

$$x_k = \int_0^T f_k(t) x(t) dt \qquad x(t) = \lim_{N\to\infty} \sum_{i=1}^{N} x_k f_k(t)$$

$$H_0: \quad x_k = s_{0k} + n_k, \quad k = 1,2,\cdots \qquad n_k = \int_0^T f_k(t) n(t) dt$$

$$H_1: \quad x_k = s_{1k} + n_k, \quad k = 1,2,\cdots \qquad s_{ik} = \int_0^T f_k(t) s_i(t) dt, \quad i = 0,1$$

判决表达式

x_k 是高斯随机过程积分的结果，因而 x_k 服从高斯分布，且有

$$E[x_k|H_0] = E\left[\int_0^T x(t) f_k(t) dt | H_0\right] = E\left[\int_0^T (s_0(t) + n(t)) f_k(t) dt\right]$$

$$= \int_0^T E[n(t)] f_k(t) dt + \int_0^T s_0(t) f_k(t) dt = s_{0k}$$

$$Var[x_k|H_0] = E[n_k^2] = E\left[\int_0^T n(t) f_k(t) dt \int_0^T n(u) f_k(u) du\right]$$

$$= \int_0^T f_k(t) \int_0^T E[n(t) n(u)] f_k(u) du dt = \frac{N_0}{2}$$

判决表达式

$$E\left[x_k \mid H_1\right] = E\left[\int_0^T x(t) f_k(t) dt \mid H_1\right] = E\left[\int_0^T (s_1(t) + n(t)) f_k(t) dt\right]$$

$$= s_{1k} + \int_0^T E[n(t)] f_k(t) dt \quad = s_{1k}$$

$$Var\left[x_k \mid H_1\right] = \frac{N_0}{2}$$

$$p\left(x_k \mid H_0\right) = \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{(x_k - s_{0k})^2}{N_0}\right) \qquad k = 1,2\cdots$$

$$p\left(x_k \mid H_1\right) = \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{(x_k - s_{1k})^2}{N_0}\right) \qquad k = 1,2\cdots$$

判决表达式

步骤2，利用前N项展开系数，构建似然比检验

由于信道是加性高斯白噪声，由卡亨南-洛维展开可知，各展开系数是不相关的，因而也是相互独立的。

$$p(\boldsymbol{x}_N|H_0)=\prod_{k=1}^{N}\frac{1}{\sqrt{\pi N_0}}\exp\left(-\frac{(x_k-s_{0k})^2}{N_0}\right)$$

$$p(\boldsymbol{x}_N|H_1)=\prod_{k=1}^{N}\frac{1}{\sqrt{\pi N_0}}\exp\left(-\frac{(x_k-s_{1k})^2}{N_0}\right)$$

$$\mathbf{x}_N=(x_1,x_2,\cdots,x_N)^T$$

判决表达式

由贝叶斯检测准则，得到

$$\Lambda(\boldsymbol{x}_N)=\frac{p(\boldsymbol{x}_N|H_1)}{p(\boldsymbol{x}_N|H_0)}\underset{H_0}{\overset{H_1}{\gtrless}}\eta$$

$$\frac{\prod_{k=1}^{N}\frac{1}{\sqrt{\pi N_0}}\exp\left(-\frac{(x_k-s_{1k})^2}{N_0}\right)}{\prod_{k=1}^{N}\frac{1}{\sqrt{\pi N_0}}\exp\left(-\frac{(x_k-s_{0k})^2}{N_0}\right)}\underset{H_0}{\overset{H_1}{\gtrless}}\eta$$

$$\exp\left(\sum_{k=1}^{N}\frac{(x_k-s_{0k})^2-(x_k-s_{1k})^2}{N_0}\right)\underset{H_0}{\overset{H_1}{\gtrless}}\eta$$

$$\frac{2}{N_0}\sum_{k=1}^{N}x_k s_{1k}-\frac{2}{N_0}\sum_{k=1}^{N}x_k s_{0k}+\frac{1}{N_0}\sum_{k=1}^{N}s_{0k}^2-\frac{1}{N_0}\sum_{k=1}^{N}s_{1k}^2\underset{H_0}{\overset{H_1}{\gtrless}}\ln\eta$$

判决表达式

步骤3，令 $N \to \infty$

$$\lim_{N\to\infty}\left(\frac{2}{N_0}\sum_{k=1}^{N}x_k s_{1k}-\frac{2}{N_0}\sum_{k=1}^{N}x_k s_{0k}+\frac{1}{N_0}\sum_{k=1}^{N}s_{0k}^2-\frac{1}{N_0}\sum_{k=1}^{N}s_{1k}^2\right)\overset{def}{=}\ln\Lambda\big(x(t)\big)$$

$$\ln\Lambda\big(x(t)\big)\underset{H_0}{\overset{H_1}{\gtrless}}\ln\eta$$

$$\frac{2}{N_0}\lim_{N\to\infty}\sum_{k=1}^{N}x_k s_{ik}=\frac{2}{N_0}\lim_{N\to\infty}\sum_{k=1}^{N}x_k\int_0^T s_i(t)f_k(t)dt$$

$$=\frac{2}{N_0}\int_0^T s_i(t)\lim_{N\to\infty}\sum_{k=1}^{N}x_k f_k(t)dt=\frac{2}{N_0}\int_0^T s_i(t)x(t)dt$$

判决表达式

$$\frac{1}{N_0}\lim_{N\to\infty}\sum_{k=1}^{N} s_{0k}^2 = \frac{1}{N_0}\int_0^T s_0^2(t)dt = \frac{E_0}{N_0}$$

$$\frac{1}{N_0}\lim_{N\to\infty}\sum_{k=1}^{N} s_{1k}^2 = \frac{1}{N_0}\int_0^T s_1^2(t)dt = \frac{E_1}{N_0}$$

$$\lim_{N\to\infty}\left(\frac{2}{N_0}\sum_{k=1}^{N} x_k s_{1k} - \frac{2}{N_0}\sum_{k=1}^{N} x_k s_{0k} + \frac{1}{N_0}\sum_{k=1}^{N} s_{0k}^2 - \frac{1}{N_0}\sum_{k=1}^{N} s_{1k}^2\right) \stackrel{def}{=} \ln\Lambda(x(t))$$

$$= \frac{2}{N_0}\int_0^T s_1(t)x(t)dt - \frac{2}{N_0}\int_0^T s_0(t)x(t)dt + \frac{1}{N_0}E_0 - \frac{1}{N_0}E_1$$

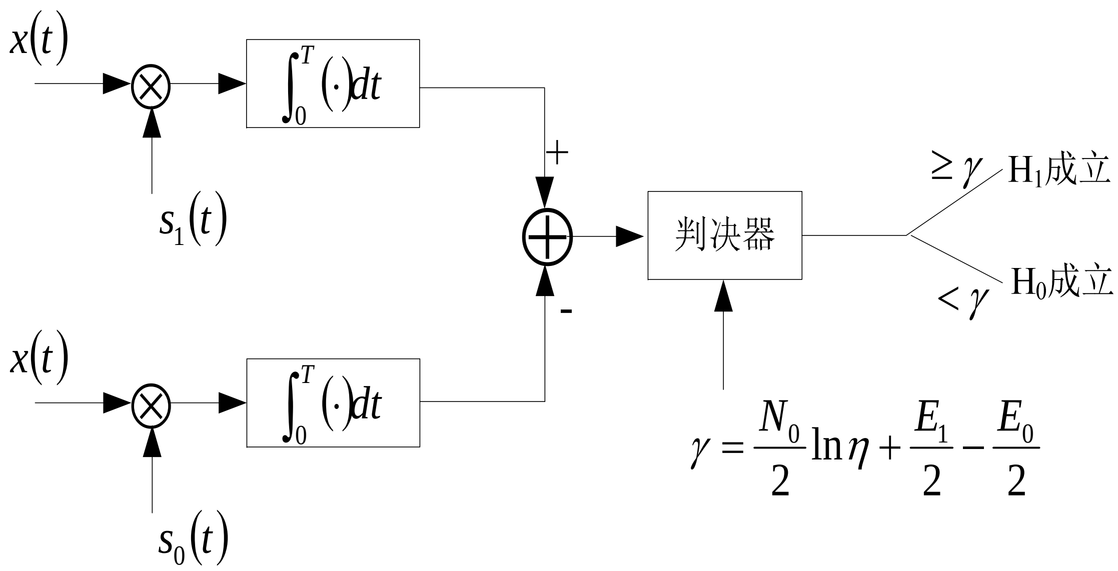
判决表达式

$$\frac{2}{N_0}\int_0^T s_1(t)x(t)sdt-\frac{2}{N_0}\int_0^T s_0(t)x(t)sd+\frac{E_0}{N_0}-\frac{E_1}{N_0}\underset{H_0}{\overset{H_1}{\gtrless}}\ln\eta$$

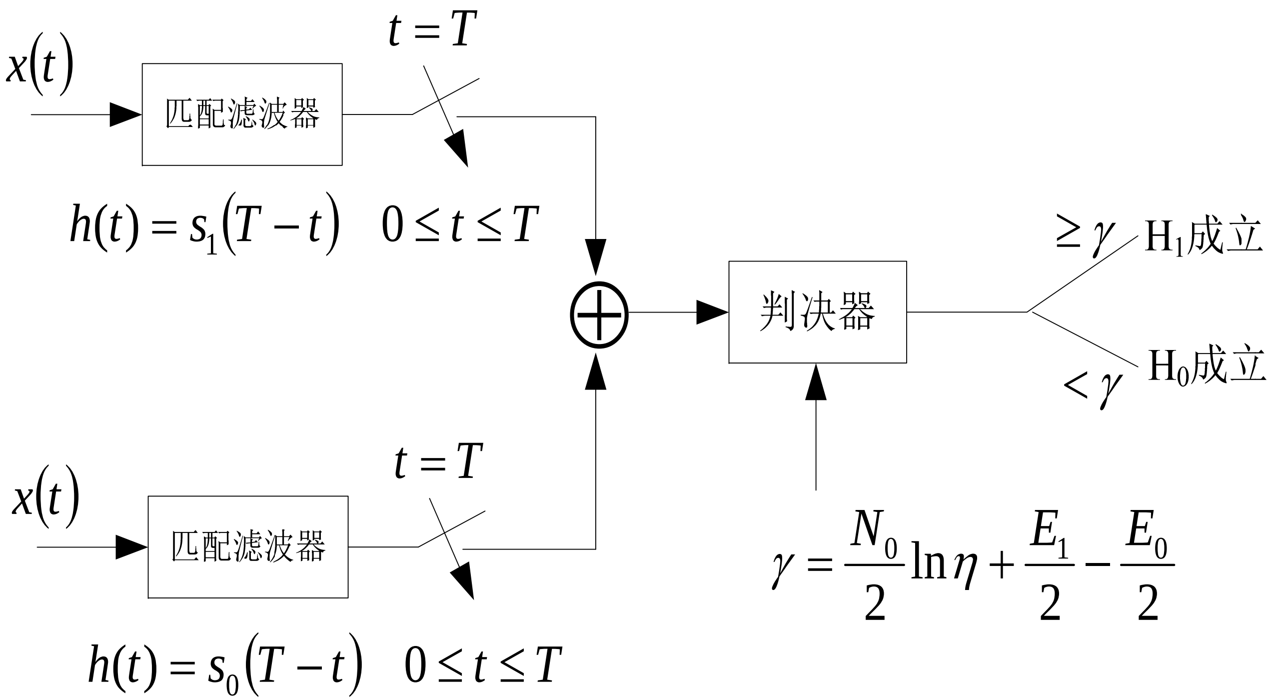
$$\int_0^T s_1(t)x(t)dt-\int_0^T s_0(t)x(t)dt\underset{H_0}{\overset{H_1}{\gtrless}}\frac{N_0}{2}\ln\eta+\frac{E_1}{2}-\frac{E_0}{2}$$

检测系统结构

$$\int_0^T s_1(t)x(t)dt-\int_0^T s_0(t)x(t)dt \underset{H_0}{\overset{H_1}{\gtrless}} \frac{N_0}{2}\ln\eta+\frac{E_1}{2}-\frac{E_0}{2}$$



检测系统结构



$x(t)$
匹配滤波器
$t=T$
$h(t)=s_1(T-t)\quad 0\le t\le T$
$x(t)$
匹配滤波器
$t=T$
$h(t)=s_0(T-t)\quad 0\le t\le T$
判决器
$\ge\gamma$ H_1成立
$<\gamma$ H_0成立
$\gamma=\frac{N_0}{2}\ln\eta+\frac{E_1}{2}-\frac{E_0}{2}$

检测性能分析

判决表达式

$$\int_0^T s_1(t)x(t)dt-\int_0^T s_0(t)x(t)dt \underset{H_0}{\overset{H_1}{\gtrless}} \frac{N_0}{2}\ln\eta+\frac{E_1}{2}-\frac{E_0}{2}$$

充分统计量 $l=\int_0^T s_1(t)x(t)dt-\int_0^T s_0(t)x(t)dt$　　门限 $\gamma=\frac{N_0}{2}\ln\eta+\frac{E_1}{2}-\frac{E_0}{2}$

$$l \underset{H_0}{\overset{H_1}{\gtrless}} \gamma$$

$$P(H_1|H_0)=\int_\gamma^\infty p(l|H_0)dl \qquad P(H_0|H_1)=\int_{-\infty}^\gamma p(l|H_1)dl$$

由于接收信号x(t)是以高斯随机过程，所以统计量l为服从高斯分布的随机变量

检测性能分析

定义统计量

$$l=\int_0^T s_1(t)x(t)dt-\int_0^T s_0(t)x(t)dt$$

$$E[l|H_0]=E\left[\int_0^T x(t)s_1(t)dt-\int_0^T x(t)s_0(t)dt\middle|H_0\right]$$

$$=E\left[\int_0^T (s_0(t)+n(t))s_1(t)dt-\int_0^T (s_0(t)+n(t))s_0(t)dt\middle|H_0\right]$$

$$=\int_0^T s_0(t)s_1(t)dt-\int_0^T s_0^2(t)dt \quad =\rho\sqrt{E_1E_0}-E_0$$

波形相关系数

$$\rho\overset{def}{=}\frac{\left|\langle s_0(t),s_1(t)\rangle\right|}{\|s_0(t)\|\|s_1(t)\|}=\frac{1}{\sqrt{E_1E_0}}\int_0^T s_0(t)s_1(t)dt;\quad |\rho|\le 1$$

检测性能分析

$$Var[l|H_0] = E\left\{\left[(l|H_0) - E[(l|H_0)]\right]^2\right\}$$

$$= E\left[\left(\int_0^T n(t)s_1(t)dt - \int_0^T n(t)s_0(t)dt\right)^2\right]$$

$$= E\left[\left(\int_0^T n(t)s_1(t)dt\right)^2\right] + E\left[\left(\int_0^T n(t)s_0(t)dt\right)^2\right]$$

$$-2E\left[\int_0^T n(t)s_1(t)dt\int_0^T n(u)s_0(u)du\right]$$

检测性能分析

$$E\left[\left(\int_0^T n(t)s_1(t)dt\right)^2\right]$$

$$= E\left[\int_0^T n(t)s_1(t)dt\int_0^T n(u)s_1(u)du\right]$$

$$= \int_0^T s_1(t)\int_0^T E\left[n(t)n(u)\right]s_1(u)dudt$$

$$= \int_0^T s_1(t)\int_0^T \frac{N_0}{2}\delta(t-u)s_1(u)dudt$$

$$= \frac{N_0}{2}\int_0^T s_1(t)s_1(t)dt = \frac{N_0E_1}{2}$$

检测性能分析

$$E\left[\int_0^T n(t)s_1(t)dt\int_0^T n(u)s_0(u)du\right]$$

$$=\int_0^T s_1(t)\int_0^T E\left[n(t)n(u)\right]s_0(u)dudt$$

$$=\int_0^T s_1(t)\int_0^T \frac{N_0}{2}\delta(t-u)s_0(u)dudt$$

$$=\frac{N_0}{2}\int_0^T s_1(t)s_0(t)dt=\frac{N_0}{2}\rho\sqrt{E_1E_0}$$

检测性能分析

$$Var\left[l \mid H_0\right] = E\left[\left(\int_0^T n(t)s_1(t)dt\right)^2\right] + E\left[\left(\int_0^T n(t)s_0(t)dt\right)^2\right]$$

$$-2E\left[\int_0^T n(t)s_1(t)dt\int_0^T n(u)s_0(u)du\right]$$

$$= \frac{N_0E_1}{2} + \frac{N_0E_0}{2} - 2\frac{N_0}{2}\rho\sqrt{E_1E_0}$$

$$= \frac{N_0}{2}\left(E_1 + E_0 - 2\rho\sqrt{E_1E_0}\right)$$

检测性能分析

$$E\left[l \middle| H_1\right] = E_1 - \rho\sqrt{E_1 E_0}$$

$$Var\left[l \middle| H_1\right] = \frac{N_0}{2}\left(E_1 + E_0 - 2\rho\sqrt{E_1 E_0}\right)$$

$$d^2 \stackrel{def}{=} \frac{\left(E\left(l \middle| H_1\right) - E\left(l \middle| H_0\right)\right)^2}{Var\left(l \middle| H_0\right)} = \frac{2}{N_0}\frac{\left(E_1 - \rho\sqrt{E_1 E_0} - \rho\sqrt{E_1 E_0} + E_0\right)^2}{E_1 + E_0 - 2\rho\sqrt{E_1 E_0}}$$

$$= \frac{2}{N_0}\left(E_1 + E_0 - 2\rho\sqrt{E_1 E_0}\right)$$

检测性能分析

$$P(H_1|H_0)=\int_{\gamma}^{\infty} p(l|H_0)dl=Q\left(\frac{\ln\eta}{d}+\frac{d}{2}\right)$$

$$P(H_0|H_0)=1-Q\left(\frac{\ln\eta}{d}+\frac{d}{2}\right)$$

$$P(H_1|H_1)=\int_{\gamma}^{\infty} p(l|H_1)dl=Q\left(\frac{\ln\eta}{d}-\frac{d}{2}\right)$$

$$P(H_0|H_1)=1-Q\left(\frac{\ln\eta}{d}-\frac{d}{2}\right)$$

最佳信号波形设计

Problem

$$\begin{cases} \max \quad d^2 = \dfrac{\left(E\left(l|H_1\right) - E\left(l|H_0\right)\right)^2}{Var\left(l|H_0\right)} = \dfrac{2}{N_0}\left(E_1 + E_0 - 2\rho\sqrt{E_1 E_0}\right) \\ s.t. \;\; E_0 + E_1 \le 2E_S \end{cases}$$

⇨

Solution

$$d^2 = \frac{8E_S}{N_0} \qquad E_0 = E_1 = E_S \qquad \rho = -1$$

$$s_0(t) = -s_1(t)$$

充分统计量法

正交级数展开法：信道噪声是白噪声，正交函数集可任意选取。

充分统计量法：选取特定的正交函数集，使得有关发送信号的信息只包含在有限的展开系数中。

充分统计量法

$$H_0: \quad x(t) = s_0(t) + n(t), \quad 0 \le t \le T$$
$$H_1: \quad x(t) = s_1(t) + n(t), \quad 0 \le t \le T$$

$$\rho = \frac{1}{\sqrt{E_0 E_1}} \int_0^T s_0(t) s_1(t) dt$$

充分统计量法

步骤1，选择一组完备正交函数集 $\{f_k(t), k=1,2\cdots\}$
满足以下条件：

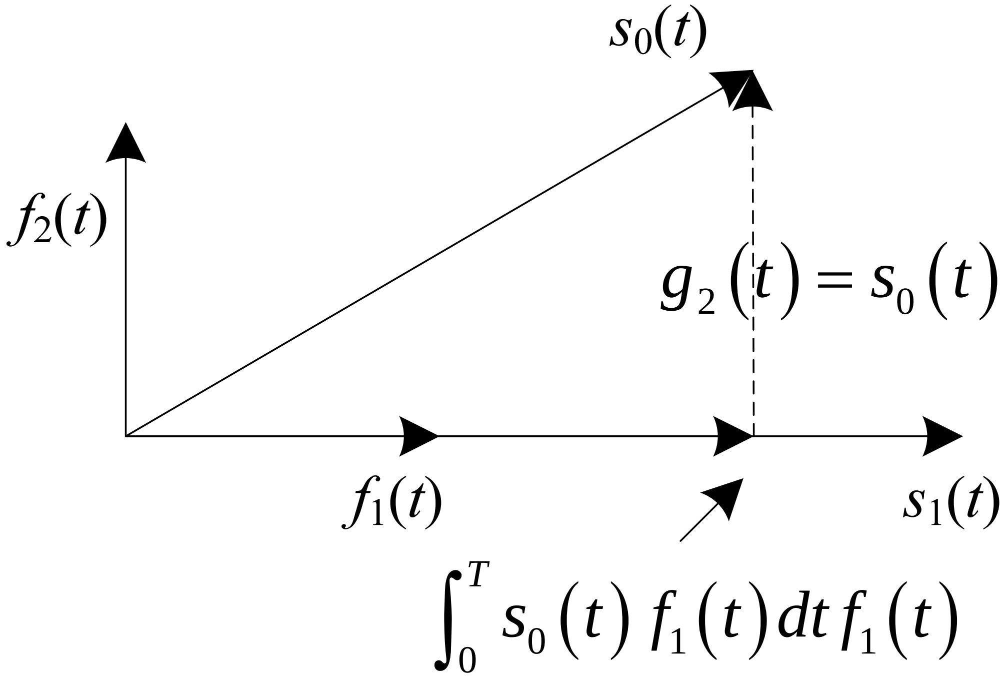
$$f_1(t)=\frac{1}{\sqrt{E_1}}s_1(t)$$

$$g_2(t)=s_0(t)-\int_0^T s_0(t)f_1(t)dt\,f_1(t)$$
$$=s_0(t)-\rho\sqrt{E_0}f_1(t)$$

$$f_2(t)=\frac{g_2(t)}{\sqrt{\int_0^T g_2^2(t)dt}}=\frac{1}{\sqrt{(1-\rho^2)E_0}}\left(s_0(t)-\rho\sqrt{\frac{E_0}{E_1}}s_1(t)\right)$$

$$\int_0^T g_2^2(t)dt=\int_0^T\left(s_0(t)-\rho\sqrt{E_0}f_1(t)\right)^2dt$$
$$=\int_0^T(s_0(t))^2dt+\rho^2E_0\int_0^T(f_1(t))^2dt-2\rho\sqrt{E_0}\int_0^T s_0(t)f_1(t)dt$$
$$=E_0+\rho^2E_0-2\rho\sqrt{E_0}\times\rho\sqrt{E_0}$$
$$=E_0-\rho^2E_0$$

充分统计量法



$$g_2(t) = s_0(t) - \int_0^T s_0(t) f_1(t) dt f_1(t)$$

充分统计量法

$f_k(t)$是分别与$f_1(t)$和$f_2(t)$正交的单位能量函数， $k \geq 3$

任意两个函数$f_k(t)$和$f_j(t)$是正交的， $k \geq 1$， $j \geq 1, k \neq j$

步骤2，利用选择的正交函数集 $\{f_k(t), k = 1,2\cdots\}$
对接收信号进行正交级数展开。

$$H_0: \quad x_1 = \int_0^T x(t) f_1(t) dt$$

$$= \int_0^T \left(s_0(t) + n(t)\right) \frac{1}{\sqrt{E_1}} s_1(t) dt$$

$$= \rho\sqrt{E_0} + n_1$$

充分统计量法

$$x_2=\int_0^T x(t)f_2(t)dt=\int_0^T\left(s_0(t)+n(t)\right)f_2(t)dt$$

$$=\int_0^T\left(s_0(t)+n(t)\right)\frac{1}{\sqrt{\left(1-\rho^2\right)E_0}}\left(s_0(t)-\rho\sqrt{\frac{E_0}{E_1}}s_1(t)\right)dt$$

$$=\frac{1}{\sqrt{\left(1-\rho^2\right)E_0}}\int_0^T\left(s_0(t)+n(t)\right)\left(s_0(t)-\rho\sqrt{\frac{E_0}{E_1}}s_1(t)\right)dt$$

$$=\frac{1}{\sqrt{\left(1-\rho^2\right)E_0}}\left[E_0-\rho\sqrt{\frac{E_0}{E_1}}\rho\sqrt{E_1E_0}\right]+n_2$$

$$=\sqrt{\left(1-\rho^2\right)E_0}+n_2$$

$$x_k=\int_0^T x(t)f_k(t)dt=n_k \qquad k\geq 3$$

充分统计量法

$$H_1: \quad x_1 = \int_0^T x(t) f_1(t) dt = \int_0^T \left(s_1(t) + n(t)\right) \frac{1}{\sqrt{E_1}} s_1(t) dt$$

$$= \sqrt{E_1} + n_1$$

$$x_2 = \int_0^T x(t) f_2(t) dt = \int_0^T \left(s_1(t) + n(t)\right) f_2(t) dt \quad = n_2$$

$$x_k = \int_0^T x(t) f_k(t) dt = n_k \qquad k \geq 3$$

充分统计量法

步骤3，利用得到的展开系数，构建似然比表达式

$$E\left[x_1|H_0\right]=E\left[\rho\sqrt{E_0}+n_1\right]=\rho\sqrt{E_0}$$

$$E\left[x_2|H_0\right]=E\left[\sqrt{\left(1-\rho^2\right)E_0}+n_2\right]=\sqrt{\left(1-\rho^2\right)E_0}$$

$$E\left[x_k|H_0\right]=E\left[n_k\right]=0\quad k\geq 3\qquad Var\left[x_k|H_0\right]=E\left[n_k^2\right]=\frac{N_0}{2}$$

$$E\left[x_1|H_1\right]=E\left[\sqrt{E_1}+n_1\right]=\sqrt{E_1}\qquad E\left[x_k|H_1\right]=E\left[n_k\right]=0\quad k\geq 2$$

$$Var\left[x_k|H_1\right]=E\left[n_k^2\right]=\frac{N_0}{2}$$

充分统计量法

$$\mathbf{x}=\left(x_1,x_2,\cdots,x_N,\cdots\right)^T$$

当k大于2时，展开系数x_k是仅与信道噪声有关的随机变量，

与发送信号检测无关，因此可以利用前两个展开系数构建贝叶斯检测，即

$$\Lambda(\boldsymbol{x})=\frac{p(\boldsymbol{x}|H_1)}{p(\boldsymbol{x}|H_0)}\underset{H_0}{\overset{H_1}{\gtrless}}\eta$$

$$\Lambda(\boldsymbol{x})=\frac{p(x_1,x_2|H_1)}{p(x_1,x_2|H_0)}\underset{H_0}{\overset{H_1}{\gtrless}}\eta$$

$$\frac{\frac{1}{\sqrt{\pi N_0}}\exp\left(-\frac{\left(x_1-\sqrt{E_1}\right)^2}{N_0}\right)\frac{1}{\sqrt{\pi N_0}}\exp\left(-\frac{x_2^2}{N_0}\right)}{\frac{1}{\sqrt{\pi N_0}}\exp\left(-\frac{\left(x_1-\rho\sqrt{E_0}\right)^2}{N_0}\right)\frac{1}{\sqrt{\pi N_0}}\exp\left(-\frac{\left(x_2-\sqrt{\left(1-\rho^2\right)E_0}\right)^2}{N_0}\right)}\underset{H_0}{\overset{H_1}{\gtrless}}\eta$$

充分统计量法

两边取对数，并化简，得

$$\exp\left(\frac{1}{N_0}\left[\left(x_1-\rho\sqrt{E_0}\right)^2+\left(x_2-\sqrt{\left(1-\rho^2\right)E_0}\right)^2\right]-\left[\left(x_1-\sqrt{E_1}\right)^2+x_2^2\right]\right)\underset{H_0}{\overset{H_1}{\gtrless}}\eta$$

$$\frac{1}{N_0}\left[2\sqrt{E_1}x_1-2\rho\sqrt{E_0}x_1-2\sqrt{\left(1-\rho^2\right)E_0}x_2-E_1+E_0\right]\underset{H_0}{\overset{H_1}{\gtrless}}\ln\eta$$

$$\left(\sqrt{E_1}-\rho\sqrt{E_0}\right)x_1-\sqrt{\left(1-\rho^2\right)E_0}x_2\underset{H_0}{\overset{H_1}{\gtrless}}\frac{N_0}{2}\ln\eta+\frac{1}{2}\left(E_1-E_0\right)$$

充分统计量法

$$l[x(t)]=\left(\sqrt{E_1}-\rho\sqrt{E_0}\right)x_1-\sqrt{\left(1-\rho^2\right)E_0}\,x_2$$

$$=\left(\sqrt{E_1}-\rho\sqrt{E_0}\right)\int_0^T x(t)\frac{1}{\sqrt{E_1}}s_1(t)\,dt$$

$$-\sqrt{\left(1-\rho^2\right)E_0}\int_0^T x(t)\frac{1}{\sqrt{\left(1-\rho^2\right)E_0}}\left(s_0(t)-\rho\sqrt{\frac{E_0}{E_1}}s_1(t)\right)dt$$

$$=\left(\sqrt{E_1}-\rho\sqrt{E_0}\right)\int_0^T x(t)\frac{1}{\sqrt{E_1}}s_1(t)\,dt-\int_0^T x(t)\left(s_0(t)-\rho\sqrt{\frac{E_0}{E_1}}s_1(t)\right)dt$$

$$=\int_0^T x(t)s_1(t)\,dt-\int_0^T x(t)s_0(t)\,dt$$

$$\int_0^T s_1(t)x(t)\,dt-\int_0^T s_0(t)x(t)\,dt\ \underset{H_0}{\overset{H_1}{\gtrless}}\ \frac{N_0}{2}\ln\eta+\frac{E_1}{2}-\frac{E_0}{2}$$

例1

考虑发送信号周期为$T=2\pi/\omega_0$ 的二元移频键控系统,在假设H_1和H_0下的发送信号分别为:

$$H_0:\quad x(t)=a\sin\omega_0 t+n(t),\qquad 0\le t\le T$$
$$H_1:\quad x(t)=a\sin 2\omega_0 t+n(t),\qquad 0\le t\le T$$

其中, 信号的振幅和频率已知,并假定两个假设先验等概.信号在传输中叠加了均值为零,功率谱密度为N0/2的高斯白噪声.现采用最小平均错误概率准则，设计信号检测系统,并计算平均错误概率.

解： 根据题设，得到两个信号的能量分别为

$$E_0=\int_0^T (a\sin\omega_0 t)^2 dt=\frac{a^2T}{2}$$

$$E_1=\int_0^T (a\sin 2\omega_0 t)^2 dt=\frac{a^2T}{2}$$

由于两个假设先验等概，因此在最小平均错误概率准则下，判决门限 $\eta=1$

利用一般二元信号检测波形判决表达式，得

$$\int_0^T s_1(t)x(t)dt-\int_0^T s_0(t)x(t)dt \underset{H_0}{\overset{H_1}{\gtrless}} \frac{N_0}{2}\ln\eta+\frac{E_1}{2}-\frac{E_0}{2}=0$$

$$\int_0^T x(t)a\sin 2\omega_0 tdt-\int_0^T x(t)a\sin\omega_0 tdt \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

为求平均错误概率，首先需要计算偏移系数

$$d^2 \overset{def}{=} \frac{\left(E(l|H_1)-E(l|H_0)\right)^2}{Var(l|H_0)}$$

$$l=\int_0^T x(t)a\sin 2\omega_0 t dt-\int_0^T x(t)a\sin\omega_0 t dt$$

$$E[l|H_0]=E\left[\int_0^T x(t)a\sin 2\omega_0 t dt-\int_0^T x(t)a\sin\omega_0 t dt \middle| H_0\right]$$

$$=E\left[\int_0^T (a\sin\omega_0 t+n(t))a\sin 2\omega_0 t dt-\int_0^T (a\sin\omega_0 t+n(t))a\sin\omega_0 t dt\right]$$

$$=-\frac{a^2T}{2}=-E_0=-E_s$$

$$\sin\alpha\sin\beta=-\frac{1}{2}\left(\cos(\alpha+\beta)-\cos(\alpha-\beta)\right)$$

$$Var[l|H_0] = E\left\{\left[\left(l|H_0\right) - E\left[\left(l|H_0\right)\right]\right]^2\right\}$$

$$= E\left[\left(\int_0^T n(t) a \sin 2\omega_0 t dt - \int_0^T n(t) a \sin \omega_0 t dt\right)^2\right]$$

$$= E\left[\left(\int_0^T n(t) a \sin 2\omega_0 t dt\right)^2\right] + E\left[\left(\int_0^T n(t) a \sin \omega_0 t dt\right)^2\right]$$

$$-2E\left[\int_0^T n(t) a \sin 2\omega_0 t dt \int_0^T n(u) a \sin \omega_0 u du\right]$$

$$= \frac{N_0 E_1}{2} + \frac{N_0 E_0}{2} = N_0 E_s$$

$$E\left[l\middle|H_1\right]=E\left[\int_0^T x(t)a\sin 2\omega_0 tdt-\int_0^T x(t)a\sin \omega_1 tdt\middle|H_1\right]$$

$$=E\left[\int_0^T \left(a\sin 2\omega_0 t+n(t)\right)a\sin 2\omega_0 tdt-\int_0^T \left(a\sin 2\omega_0 t+n(t)\right)a\sin \omega_0 tdt\right]$$

$$=E_1=E_s$$

$$d^2\overset{def}{=}\frac{\left(E\left(l\middle|H_1\right)-E\left(l\middle|H_0\right)\right)^2}{Var\left(l\middle|H_0\right)}=\frac{\left(E_s-\left(-E_s\right)\right)^2}{N_0E_s}$$

$$=\frac{4E_s}{N_0}$$

$$P(H_1|H_0)=\int_\gamma^\infty p(l|H_0)dl = Q\left(\frac{\ln\eta}{d}+\frac{d}{2}\right) = Q\left(\frac{d}{2}\right)=Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

$$P(H_1|H_1)=\int_\gamma^\infty p(l|H_1)dl = Q\left(\frac{\ln\eta}{d}-\frac{d}{2}\right)$$

$$P(H_0|H_1)= 1-Q\left(\frac{\ln\eta}{d}-\frac{d}{2}\right) = 1-Q\left(-\frac{d}{2}\right)=1-Q\left(-\sqrt{\frac{E_s}{N_0}}\right)$$

$$= Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

$$P_e = P(H_0)P(H_1|H_0)+P(H_1)P(H_0|H_1)=Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

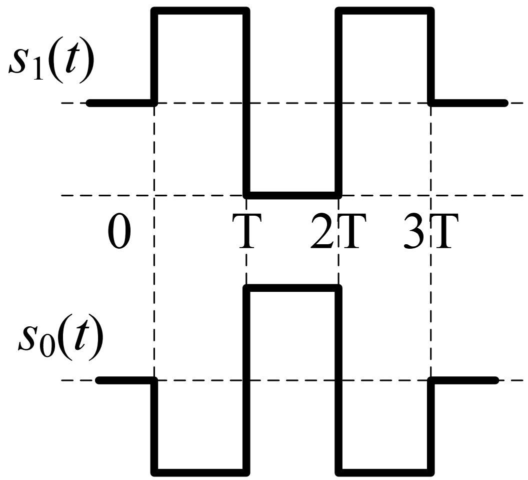
例2

- 在二元数字通信系统中，两个假设下的接收信号分别为

$$\begin{cases} H_0 : x(t) = s_0(t) + n(t), 0 \le t \le 3T \\ H_1 : x(t) = s_1(t) + n(t), 0 \le t \le 3T \end{cases}$$

其中，信号$s_1(t)$和$s_0(t)$波形如右图所示，加性噪声是均值为0，功率谱密度为$N_0/2$的AWGN；设信号先验等概，采用MAEP求E_s/N_0=2时的平均错误概率Pe,其中Es是的平均能量，即

$$E_S = \frac{1}{2}\left[\int_0^{3T} s_0^2(t)dt + \int_0^{3T} s_1^2(t)dt\right]$$



解： 根据题设，得到两个信号的能量分别为

$$E_0=\int_0^{3T}\left(s_0(t)\right)^2 dt \qquad E_1=\int_0^{3T}\left(s_1(t)\right)^2 dt$$

由于两个假设先验等概，因此在最小平均错误概率准则下，判决门限 $\eta=1$

利用一般二元信号检测波形判决表达式，得

$$\int_0^{3T}s_1(t)x(t)dt-\int_0^{3T}s_0(t)x(t)dt\underset{H_0}{\overset{H_1}{\gtrless}}\frac{N_0}{2}\ln\eta+\frac{E_1}{2}-\frac{E_0}{2}$$

由于 $s_1(t)=-s_0(t)$

所以 $E_0=E_1=E_s$

$$\int_0^{3T}s_1(t)x(t)dt\underset{H_0}{\overset{H_1}{\gtrless}}0$$

为求平均错误概率，首先需要计算偏移系数

$$d^2 \overset{def}{=} \frac{\left(E(l|H_1)-E(l|H_0)\right)^2}{Var(l|H_0)}$$

在两种假设下，统计量 $l=\int_0^{3T}x(t)s_1(t)dt$ 均是高斯随机变量，因此有

$$E[l|H_0]=E\left[\int_0^{3T}x(t)s_1(t)dt|H_0\right]$$

$$=E\left[\int_0^{3T}(s_0(t)+n(t))s_1(t)dt\right]$$

$$=-E_s$$

$$Var[l|H_0] = E\left\{\left[(l|H_0) - E[(l|H_0)]\right]^2\right\}$$

$$= E\left[\left(\int_0^{3T} n(t)s_1(t)dt\right)^2\right] = \frac{N_0 E_s}{2}$$

$$E[l|H_1] = E\left[\int_0^{3T} x(t)s_1(t)dt | H_1\right]$$

$$= E\left[\int_0^{3T} (s_1(t) + n(t))s_1(t)dt\right]$$

$$= E_s$$

$$Var[l|H_1] = E\{[(l|H_1) - E[(l|H_1)]]^2\}$$

$$= E\left[\left(\int_0^{3T} n(t)s_1(t)dt\right)^2\right] = \frac{N_0 E_s}{2}$$

$$d^2 \overset{def}{=} \frac{\left(E(l|H_1) - E(l|H_0)\right)^2}{Var(l|H_0)} = \frac{\left(E_s - (-E_s)\right)^2}{N_0 E_s / 2}$$

$$= \frac{8E_s}{N_0}$$

$$P(H_1|H_0)=\int_{\gamma}^{\infty} p(l|H_0)dl \quad = Q\left(\frac{\ln\eta}{d}+\frac{d}{2}\right) = Q\left(\frac{d}{2}\right) = Q\left(\sqrt{\frac{2E_s}{N_0}}\right)$$

$$P(H_1|H_1)=\int_{\gamma}^{\infty} p(l|H_1)dl = Q\left(\frac{\ln\eta}{d}-\frac{d}{2}\right)$$

$$P(H_0|H_1)= \quad 1-Q\left(\frac{\ln\eta}{d}-\frac{d}{2}\right)=1-Q\left(-\frac{d}{2}\right)=1-Q\left(-\sqrt{\frac{2E_s}{N_0}}\right)$$

$$=Q\left(\sqrt{\frac{2E_s}{N_0}}\right)$$

$$P_e = P(H_0)P(H_1|H_0)+P(H_1)P(H_0|H_1)=Q\left(\sqrt{\frac{2E_s}{N_0}}\right)$$