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Short communication

Exact thresholds for low-density parity-check codes over the binary erasure channel

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Abstract

A simple method for determining the threshold of irregular LDPC codes over the binary erasure channel (BEC) under message-passing decoding is proposed. An exact formula for calculating the threshold of irregular LDPC codes over the BEC is proved. This generalizes the known result on the threshold of regular LDPC codes to irregular LDPC codes. Moreover, our new method can avoid the computation of the inverse of the degree distribution function for irregular LDPC codes. Numerical results demonstrate the correctness of our proposed method.

Keywords: Binary erasure channel (BEC); LDPC codes; Threshold

1. Introduction

The binary erasure channel (BEC), presented by Elias in 1955 [1], has lately become increasingly popular, as it can be used to model Internet transmission systems. Low-density parity-check (LDPC) codes over the BEC have been studied extensively lately [2–12].

LDPC codes can be characterized by a bipartite graph between a set of left nodes called variable nodes and a set of right nodes called check nodes. A degree distribution for the graph is the pair (λ, ρ), where $\lambda(x)$ and $\rho(x)$ are functions of the form

$$\lambda(x) = \sum_{i=2}^{\ell} \lambda_i x^{i-1}, \quad \rho(x) = \sum_{i=2}^{r} \rho_i x^{i-1}$$

and $\lambda_i(\rho_i)$ denote(s) the fraction of edges in the bipartite graph that are connected to a left (right) node of degree i [2,3]. If $\lambda(x) = x^{d_v-1}$ and $\rho(x) = x^{d_c-1}$, then the code is said $(d_v, d_c)-$ regular, otherwise irregular. In Ref. [2], Luby et al. presented an iterative message-passing (MP) algorithm for decoding LDPC codes over the BEC and showed that the proposed decoding algorithm is successful on a random graph with degree distribution (λ, ρ) and initial erasure probability δ if

$$\delta\lambda(1 - \rho(1 - x)) < x \tag{1}$$

on the interval $(0, \delta)$ [2].

Density evolution for the BEC can be expressed as follows. For a given degree distribution (λ, ρ), the expected fraction of erasure messages emitted in the n-th iteration, call it x_n, is given by the recursion

$$x_n = x_n(\delta) = \delta\lambda(1 - \rho(1 - x_{n-1})) \tag{2}$$

where $n \geqslant 1$ and x_0, the initial fraction of erasures is equal to the erasure probability δ of the channel [3]. We are interested in the supremum of $\delta, \delta < 1$, such that $x_n(\delta)$ converges to zero as n tends to infinity. The largest value of δ satisfying (1) for the given parameter pair (λ, ρ) is called

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the threshold and denoted by $\delta^*(\lambda,\rho)$ [3]. That is to say, the threshold $\delta^*(\lambda,\rho)$ associated with the degree distribution (λ,ρ) is defined as

$$\delta^*(\lambda,\rho) = \sup\left\{\delta|0<\delta<1, \lim_{n\to\infty} x_n(\delta)=0, x_0=\delta\right\}$$

Some results on the threshold of LDPC codes over the BEC under MP decoding have been achieved [8–11]. In Refs. [8,9], an exact formula of the threshold for $(d_v, d_c)-$ regular LDPC codes over the BEC under MP decoding was presented. This exact formula was given by

$$\delta^*(\lambda,\rho) = \min_{x\in[0,1]}\{f_1(x)\}$$

where $f_1(x)=(1-x)(1-x^{d_c-1})^{d_v-1}$. In Ref. [11], by analyzing EXIT (Extrinsic Information Transfer) charts, Hehn et al. presented a method for determining the threshold of irregular LDPC codes over the BEC under MP decoding. However, the disadvantage of this method is that it is necessary to compute the inverse of the degree distribution function used for determining the code threshold. Moreover, it is impossible to solve this inverse function for general irregular LDPC codes over the BEC.

In this paper, we propose a simple method for determining the threshold of irregular LDPC codes over the BEC under the MP decoding algorithm. By generalizing the results of the threshold of regular LDPC codes presented in Refs. [8,9] to irregular LDPC codes, we show an exact formula for calculating the threshold of general LDPC codes over the BEC. The proposed method does not require the computation of the inverse of the degree distribution function for irregular LDPC codes. Finally, some numerical evidence is provided to show the correctness of this formula.

2. Lemmas

As stated before, if $\delta\lambda(1-\rho(1-x))<x$ on the interval $(0,\delta)$, then the decoder of LDPC codes over the BEC under MP decoding can correct a δ fraction of erasures (losses). This condition can be translated to $\delta < x\lambda(1-\rho(1-x))$. In order to determine the threshold $\delta^*(\lambda,\rho)$ of LDPC codes over the BEC, let f be defined in $[0,1)$ by

$$f(x)=\frac{1-x}{\lambda(1-\rho(x))} \tag{3}$$

Thus we have

$$f(x)=\frac{1-x}{\sum_{i=2}^{\ell}\lambda_i\left(1-\sum_{j=2}^{r}\rho_j x^{j-1}\right)^{i-1}} = \frac{1}{\lambda_2 h(x)+\sum_{i=3}^{\ell}\lambda_i(1-x)^{i-2}[h(x)]^{i-1}}$$

where $h(x)=\sum_{i=2}^{r}\rho_i\sum_{j=0}^{i-2}x^j$.

In order to propose a simple method for determining the threshold of irregular LDPC codes over the BEC under the MP decoding algorithm, the function s is defined in [0, 1] by

$$s(x)=\frac{1}{\lambda_2 h(x)+\sum_{i=3}^{\ell}\lambda_i(1-x)^{i-2}[h(x)]^{i-1}} \tag{4}$$

In order to determine the threshold of the general class of LDPC codes, we start with the following lemmas.

Lemma 1. *For a given degree distribution (λ,ρ), let s be the function as defined above in* (4). *Then the minimum of $s(x)$ in* [0, 1] *exists.*

Proof. By the definition of s, we note that $s(x)$ is a continuous function in [0, 1]. By Weierstrass' second theorem [13], we obtain that the minimum of $s(x)$ in [0, 1] exists. □

Lemma 2. *For a given degree distribution (λ,ρ) with the erasure probability $\delta\in(0,1)$, define the function $g(\delta,x)=\delta\lambda(1-\rho(1-x))$ in* [0, 1]. *The sequence $\{x_n(\delta)\}$ is defined by $x_n=x_n(\delta)=\delta\lambda(1-\rho(1-x_{n-1}))$ as in* (2). *For every $\delta\in(0,1)$, if $g(\delta,x)<x$ for all real $x\in(0,1)$, then we have $\lim_{n\to\infty}x_n(\delta)=0$.*

Proof. Suppose for every $\delta\in(0,1)$, $g(\delta,x)<x$ for all real $x\in(0,1)$. By the definition of g and $\{x_n(\delta)\}$ we have

$$x_n=\delta\lambda(1-\rho(1-x_{n-1}))=g(\delta,x_{n-1})<x_{n-1}$$

for every integer $n\geq 1$. Note that $\lambda(0)=\rho(0)=0$ and $\lambda(1)=\rho(0)=1$. It is clear that

$$0<x_n=\delta\lambda(1-\rho(1-x_{n-1}))<\delta.$$

These two results imply that the sequence $\{x_n(\delta)\}$ is bounded and strictly decreasing for every $\delta\in(0,1)$. Thus, it follows that the limit $\lim_{n\to\infty}x_n(\delta)$ exists.

Let $\lim_{n\to\infty}x_n(\delta)=x''$ for some $x''\in(0,1)$. Since $g(\delta,x)$ is a continuous function of x for $x\in[0,1]$, we must have $g(\delta,x'')=x''$. In conjunction with $g(\delta,0)=0$, by the uniqueness of the limit x'', we obtain $x''=0$. This says $\lim_{n\to\infty}x_n(\delta)=0$ for every $\delta\in(0,1)$. □

3. Exact thresholds for LDPC codes

In this section, our main purpose is to show that the threshold $\delta^*(\lambda,\rho)$ of LDPC codes over the BEC under MP decoding is equal to the minimum of $s(x)$ in [0, 1]. Using Lemmas 1 and 2, we obtain the following theorem.

Theorem 1. *Let f and s be the functions as defined above in* (3) *and* (4), *respectively*. For LDPC codes over the BEC with the degree distribution (λ,ρ) and the initial fraction of erasures $\delta\in(0,1)$, the code threshold $\delta^*(\lambda,\rho)$ below which the MP decoding leads to vanishing erasure probability is given by

$$\delta^*(\lambda,\rho)=\min_{x\in[0,1]}\{s(x)\}.$$

Proof. Let g be the function defined in [0, 1] by $g(\delta, x) = \delta\lambda(1-\rho(1-x))$. Let

$$S = \left\{\delta | 0 < \delta < 1, \lim_{n\to\infty} x_n(\delta) = 0\right\},$$

where $\{x_n(\delta)\}$ is as defined recursively in (2). By Lemma 1, we note that $\min_{x\in[0,1]}\{s(x)\}$ exists. Thus, let $s^* = \min_{x\in[0,1]}\{s(x)\}$. Note that $f(x) = s(x)$ for all $x \in [0, 1)$.

First we prove that $\delta \le s^*$ for all $\delta \in S$. Suppose, to get a contradiction, there exists $\delta' \in S$ such that $\delta' > s^*$. By the continuity of s, we can choose $x' \in (0, 1]$ such that

$$\delta' > s(1-x') = f(1-x') = \frac{x'}{\lambda(1-\rho(1-x'))}.$$

Thus we note that $\delta'\lambda(1-\rho(1-x')) > x' > 0$. Then

$$\delta' \ge g(\delta', x') = \delta'\lambda(1-\rho(1-x')) > x' > 0 \tag{5}$$

When $x_0 = \delta'$, it follows from (5) and the fact that $g(\delta', x)$ is a strictly increasing function in its second argument for $x \in [0, 1]$ that

$$x_1(\delta') = g(\delta', x_0) = g(\delta', \delta') > g(\delta', x') > x' > 0$$

and

$$x_2(\delta') = g(\delta', x_1) > g(\delta', x') > x' > 0.$$

Note that $\lambda(0) = \rho(0) = 0$ and $\lambda(1) = \rho(0) = 1$. It is clear that $0 < x_n(\delta') = \delta'\lambda(1-\rho(1-x_{n-1})) < \delta'$. It is easy to note that

$$x_1(\delta') = g(\delta', x_0) = \delta'\lambda(1-\rho(1-x_0)) < \delta' = x_0$$

and

$$x_2(\delta') = g(\delta', x_1) < g(\delta', x_0) = x_1(\delta').$$

By induction, we note that $\{x_n(\delta')\}$ is a strictly decreasing sequence and $x_n(\delta') > x' > 0$ for all integers $n \ge 1$. These two results imply that the limit $\lim_{n\to\infty} x_n(\delta')$ exists and $\lim_{n\to\infty} x_n(\delta') \ge x' > 0$. This contradicts the fact that $\lim_{n\to\infty} x_n(\delta) = 0$ for any $\delta \in S$.

Next, we prove that for any given sufficiently small $\varepsilon > 0$, there exists $\delta'' \in S$ such that $\delta'' > s^* - \varepsilon$. Since $0 < s^* \le 1$ and $s(x)$ is a continuous function in [0, 1], we can choose sufficiently small $\varepsilon > 0$, such that $s^* - \varepsilon > 0$. Let $\delta'' = s^* - \varepsilon/2$. It is easy to note that $1 > \delta'' = s^* - \varepsilon/2 > s^* - \varepsilon > 0$ and $\delta'' \in (0, 1)$. In order to show $\delta'' \in S$, we need to only prove $\lim_{n\to\infty} x_n(\delta'') = 0$. Since $\delta'' = s^* - \varepsilon/2 < s^*$, then for every $x \in (0, 1]$, we have

$$\begin{aligned} g(\delta'', x) &= \delta''\lambda(1-\rho(1-x)) \\ &= \delta''\frac{x}{f(1-x)} < \frac{s^*}{s(1-x)}x \le x \end{aligned} \tag{6}$$

When $\delta = \delta''$, it follows from (6) and Lemma 2 that $\lim_{n\to\infty} x_n(\delta'') = 0$.

Hence it is concluded from these two results that $\sup S = s^*$ [13]. By the definition of the threshold and Lemma 1, we obtain $\delta^*(\lambda, \rho) = \min_{x\in[0,1]}\{s(x)\}$.

This completes the proof. □

Remark 1. It should be pointed out that in this theorem the conclusion $\delta^*(\lambda, \rho) = \min_{x\in[0,1]}\{s(x)\}$ cannot be replaced by $\delta^*(\lambda, \rho) = \min_{x\in[0,1]}\{f(x)\}$ since the minimum of $f(x)$ in [0, 1] $\min_{x\in[0,1]}\{f(x)\}$ may not exist. In the following we provide an example to prove the nonexistence of the minimum of $f(x)$ in [0, 1] $\min_{x\in[0,1]}\{f(x)\}$.

Now we consider irregular LDPC codes over the BEC with the following degree distribution (λ, ρ) as follows:

$$\lambda(x) = 0.4706x + 0.2353x^7 + 0.2941x^{29}$$
$$\rho(x) = 0.6864x^6 + 0.3136x^7.$$

For this (λ, ρ), the curve of $s(x)$ corresponding to the degree distribution (λ, ρ) in Theorem 1 is given in Fig. 1, which shows $x = 1$ is the point at which $s(x)$ attains its minimum in [0, 1]. Note that $f(x) = s(x)$ for all $x \in [0, 1)$. However, $x = 1$ is a pole for $f(x)$. It follows that from these two facts given above that the minimum of $f(x)$ in [0, 1) does not exist. However, the proposed generalization can solve this problem.

In fact, by applying numerical solution methods and programs like Matlab to (4) corresponding to the degree distribution (λ, ρ), we obtain that the code threshold

$$\delta^*(\lambda, \rho) = \min_{x\in[0,1]}\{s(x)\} = s(1) = 0.336567$$

whereas for LDPC codes over the BEC with this degree distribution (λ, ρ), by means of density evolution we obtain the same result.

4. Numerical results

In order to demonstrate the correctness and the exactness of the proposed formula, this section compares the results obtained from density evolution for the BEC with the results obtained from our new method. By using the binary search method we find the threshold by means of density evolution. The number of iterations for the density evolution formula (2) is 10^3 and 10^5, respectively.

Now we consider irregular LDPC codes over the BEC with the two representative degree distributions (λ', ρ') and (λ'', ρ'') proposed in Ref. [4] as follows:

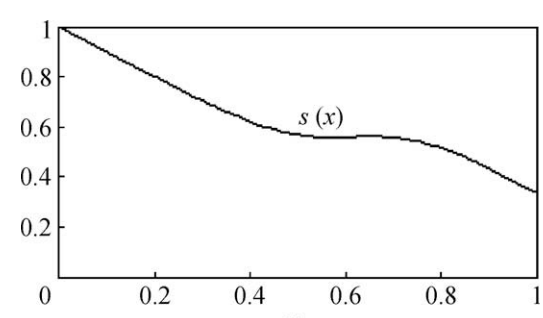

Fig. 1. The curve of $s(x)$ corresponding to the degree distribution (λ, ρ).

Table 1
Thresholds for irregular LDPC codes over the BEC, determined with density evolution and our proposed method.

Degree distribution	Threshold, density evolution		Exact threshold
	10^3 Iterations	10^5 Iterations	
(λ', ρ')	0.4361**1812275537**	0.43612820**534656**	0.43612820632214
(λ'', ρ'')	0.42057**816993587**	0.42058449**049028**	0.42058449110647

(i) $\lambda'(x) = 0.0769x + 0.6923x^2 + 0.2308x^5$

$\rho'(x) = 0.4615x^5 + 0.5385x^6$

(ii) $\lambda''(x) = 0.4706x^2 + 0.2353x^7 + 0.2941x^{29}$

$\rho''(x) = 0.7843x^9 + 0.2157x^{10}$.

By applying numerical solution methods or programs like Matlab to (4), we obtain the exact thresholds

$$\delta^*(\lambda', \rho') = 0.43612820632214$$

and

$$\delta^*(\lambda'', \rho'') = 0.42058449110647.$$

For these two degree distributions (λ', ρ') and (λ'', ρ''), by means of density evolution we obtain the approximate thresholds of the corresponding LDPC codes with the rate 1/2 over the BEC under MP decoding. Table 1 shows the results of the threshold for irregular LDPC codes over the BEC with the degree distributions (λ', ρ') and (λ'', ρ''). The last three columns in Table 1 demonstrate the correctness and the exactness of Theorem 1. Bold typed digits in Table 1 differ from the exact threshold.

For LDPC codes over the BEC with the degree distributions (λ', ρ') and (λ'', ρ''), we note that the code thresholds still differ in the magnitude of 10^{-5} when using iterations of 10^3 and 10^{-9} for 10^5 iterations. The results from density evolution could be improved by a more extensive search, but this would lead to an even higher computational complexity, whereas our numerical results given above show that our results are exact and can be calculated very quickly.

For determining the threshold of irregular LDPC codes over the BEC under MP decoding, when using the method presented by Hehn et al. in Ref. [11], we must compute the inverse of the degree distribution function $\rho(x)$. However, it is impossible to solve the inverse of this function, whereas our new method can overcome this shortcoming.

5. Conclusion

We show an exact formula for calculating the threshold of LDPC codes over the BEC under MP decoding. Thus, the result on the threshold of regular LDPC codes presented in Refs. [8,9] can be generalized to irregular LDPC codes.

Furthermore, when using the method presented by Hehn et al. in Ref. [11] for determining the threshold of irregular LDPC codes over the BEC under MP decoding, we must find the inverse of the degree distribution function used, whereas our new method can avoid this inverse operation. Numerical results demonstrate the correctness of our formula. These theoretical results on code thresholds will benefit the study of the asymptotic behavior of LDPC codes.

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References

[1] Elias P. Coding for two noisy channels. In: Proceedings of information theory. Third London symposium, London, UK: September; 1955, p. 61–74.
[2] Luby M, Mitzenmacher M, Shokrollahi MA, et al. Efficient erasure correcting codes. IEEE Trans Inform Theory 2001;47(2):569–84.
[3] Richardson T, Shokrollahi A, Urbanke R. Design of capacity approaching irregular low-density parity-check codes. IEEE Trans Inform Theory 2001;47(2):619–37.
[4] Pishro-Nik H, Fekri F. On decoding of low-density parity-check codes over the binary erasure channel. IEEE Trans Inform Theory 2004;50(3):439–54.
[5] Oswald P, Shokrollahi MA. Capacity-achieving sequences for the erasure channel. IEEE Trans Inform Theory 2002;48(12):3017–28.
[6] Richardson T, Urbanke R. The capacity of low-density parity check codes under message-passing decoding. IEEE Trans Inform Theory 2001;47(2):599–618.
[7] Guruswami V. Iterative decoding of low-density parity check codes. Bull Eur Assoc Theor Comput Sci 2006;90:53–88.
[8] Bazzi L, Richardson T, Urbanke R. Exact thresholds and optimal codes for the binary symmetric channel and Gallager's decoding algorithm A. IEEE Trans Inform Theory 2004;50(9):2010–21.
[9] Mu JJ, Wang P, Wang XM. Analysis of regular low-density erasure codes. J Xidian Univ 2003;30(4):468–72 (in Chinese).
[10] Paolini E, Chiani M. On the threshold of right regular LDPC codes for the erasure channel. IEEE Vehicular Technology Conference (VTC 2005 Spring). Stockholm, Sweden: May; 2005, p. 664–7.
[11] Hehn T, Dönmez A, Huber JB. Exact thresholds for LDPC codes transmitted over binary erasure channels. In: Proceedings of the 43rd annual allerton conference on communication, control, and computing, Monticello, USA: September; 2005, p. 43–56.
[12] Shokrollahi MA. New sequences of linear time erasure codes approaching the channel capacity. In: Proceedings of the 13th international symposium on applied algebra, algebraic algorithms, and error-correcting codes. Berlin: Springer-Verlag; 1999, 1719, p. 65–76.
[13] Nikolsky SM. A course of mathematical analysis, vol. 1. Moscow: MIR Publisher; 1977, p. 55–6.