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Entire solutions of nonlinear cellular neural networks with distributed time delays

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Abstract
The aim of this work is to study the existence of entire solutions of nonlinear cellular neural networks with distributed time delays (DCNN). The entire solutions are defined in the whole space and for all time $t \in \mathbb{R}$. From Yu *et al* (2011 *J. Diff. Eqns* **251** 630–50), we know that the DCNN model admits travelling front solutions. Combining the travelling front solutions with different wave speeds and a spatially independent solution of the DCNN model, we establish some new entire solutions to describe the interactions of travelling fronts. Various qualitative features of the entire solutions are also investigated in this work.

Mathematics Subject Classification: 34K05, 34K30, 37L60

1. Introduction

The aim of this work is to study the existence of entire solutions of nonlinear cellular neural networks with distributed time delays (DCNN). The methodology of cellular neural network (CNN) was first proposed by Chua and Yang [4–6] as an achievable alternative to fully connected neural networks in electric circuit systems. The infinite system of ordinary differential equations for CNN distributed in an one-dimensional integer lattice with a neighbourhood of radius m but without inputs can be described by

$$x_n'(t) = z - x_n(t) + \sum_{i=1}^{m} \alpha_i f(x_{n-i}(t)) + af(x_n(t)) + \sum_{j=1}^{m} \beta_j f(x_{n+j}(t)), \qquad (1.1)$$

where $n \in \mathbb{Z}$, $t \in \mathbb{R}$, $m \in \mathbb{N}$, the real coefficients a, α_i, β_i with $\sum(\alpha_i^2 + \beta_i^2) \neq 0$ of the output function f constitute the so-called space-invariant template that measure the synaptic weights of self-feedback and neighbourhood interactions. The quantity z is called a threshold or bias term and is related to the independent voltage sources in electric circuits. For more details on the circuit diagram and connection pattern for practical applications, we refer to [4–6] and the references cited therein.

In past decades, there have been extensive investigations on neural network systems, which are applied to a broad scope of fields such as image and video signal processing, robotic and biological versions and higher brain functions (see [34, 43] for more details). In particular, if the output function f is a piecewise-linear function defined by

$$f(x) = \tfrac{1}{2}(|x+1| - |x-1|) \qquad \text{for } x \in \mathbb{R}, \tag{1.2}$$

some incisive mathematical analyses have been done in [17–22] and many references cited therein. However, in view of the finite switching speed and finite velocity of signal transmission, time delay should be considered in the CNN systems. For example, Hsu *et al* [17] considered the following delay CNN (DCNN for short) system

$$\begin{aligned} x_n'(t) = &-x_n(t) + \sum_{i=1}^{m} \alpha_i \int_0^{\tau} k(s) f(x_{n-i}(t-s))\,\mathrm{d}s + a f(x_n(t)) \\ &+ \sum_{j=1}^{m} \beta_j \int_0^{\tau} k(s) f(x_{n+j}(t-s))\,\mathrm{d}s, \qquad n \in \mathbb{Z}, \quad t \in \mathbb{R}, \end{aligned} \tag{1.3}$$

where $\tau > 0$, $k : [0, \tau] \to [0, +\infty)$ is a prescribed piecewise continuous function which satisfies $\int_0^{\tau} k(s)\,\mathrm{d}s = 1$. Such function $k(\cdot)$ is called the density function for delay effect. In [17], the authors investigated the diversity of travelling wave solutions of (1.3) with the output function (1.2). More precisely, using the monotone iteration scheme, they proved the existence of monotone travelling waves provided the templates satisfy the so-called quasi-monotonicity condition, i.e. $\alpha_i \geqslant 0$, $\beta_i \geqslant 0$ for $i = 1, \ldots, m$ and $\sum(\alpha_i^2 + \beta_i^2) \neq 0$. Moreover, they considered two special cases of (1.3) in which each cell interacts only with either the nearest m left neighbours or the nearest m right neighbours. For the former case, the analytic solution in an explicit form was directly figured out. For the latter case, the deformation of travelling wave solutions with respect to the wave speed was clarified.

Recently, Liu *et al* [29] considered the existence of monotone travelling waves for the following DCNN model with a *nonlinear* output function:

$$x_n'(t) = -x_n(t) + \alpha \int_0^{\tau} k_1(s) f(x_n(t-s))\,\mathrm{d}s + \beta \int_0^{\tau} k_2(s) f(x_{n+1}(t-s))\,\mathrm{d}s, \tag{1.4}$$

where $k_i : [0, \tau] \to [0, +\infty)$ is a prescribed piecewise continuous function satisfying $\int_0^{\tau} k_i(s)\,\mathrm{d}s = 1$, $i = 1, 2$ and f is nonlinear, non-decreasing and odd on $[-1, 1]$. For example, the output function $f(x)$ can be given by

$$f(x) = \begin{cases} 1, & \text{if } x \geqslant 1, \\ \sin\dfrac{\pi}{2}x, & \text{if } |x| \leqslant 1, \\ -1, & \text{if } x \leqslant -1 \end{cases} \tag{1.5}$$

and

$$f(x) = \begin{cases} 1, & \text{if } x \geqslant 1, \\ 2x - x^2, & \text{if } 0 \leqslant x \leqslant 1, \\ 2x + x^2, & \text{if } -1 \leqslant x \leqslant 0, \\ -1, & \text{if } x \leqslant -1, \end{cases} \tag{1.6}$$

respectively.

More recently, Yu *et al* [48] extended the existence results of monotone travelling waves in [17, 18, 20, 29, 42] to a more general DCNN model of the form

$$x_n'(t) = -x_n(t) + \sum_{i=1}^{m} \alpha_i \int_0^\tau J_i(s) f(x_{n-i}(t-s))\,\mathrm{d}s + a \int_0^\tau J_{m+1}(s) f(x_n(t-s))\,\mathrm{d}s$$
$$+ \sum_{j=1}^{\ell} \beta_j \int_0^\tau J_{m+1+j}(s) f(x_{n+j}(t-s))\,\mathrm{d}s, \qquad n \in \mathbb{Z}, \quad t \in \mathbb{R}, \tag{1.7}$$

where $m, \ell \in \mathbb{N}$, $\tau > 0$, $a \geqslant 0$, $\alpha_i \geqslant 0$, $i = 1, \cdots, m$ and $\beta_j \geqslant 0$, $j = 1, \cdots, \ell$ are given constants with $\sum_{i=1}^{m} \alpha_i + \sum_{j=1}^{\ell} \beta_j > 0$; the output function f and the density function J_i, $i = 1, \cdots, m + \ell + 1$ satisfy the following assumptions:

(A1) $f \in C([0, \infty), [0, \infty))$, $f(0) = 0$, $f''(0)$ exists,

$$(a + \alpha + \beta) f(K) = K \qquad \text{and} \qquad (a + \alpha + \beta) f(u) > u \qquad \text{for } u \in (0, K),$$

where $K > 0$ is a constant, $\alpha = \sum_{i=1}^{m} \alpha_i$ and $\beta = \sum_{j=1}^{\ell} \beta_j$;

(A2) $f(u)$ is non-decreasing for $u \in [0, K]$ such that

$$af'(0) \geqslant 1 \quad \text{and} \quad |f(u) - f(v)| \leqslant f'(0)|u - v| \qquad \text{for all} \qquad u, v \in [0, K];$$

(A3) $J_i \in L^1([0, \tau])$ is a non-negative function satisfying

$$\int_0^\tau J_i(s)\,\mathrm{d}s = 1, \qquad \text{for } i = 1, \ldots, m + \ell + 1.$$

It is obvious that the output functions defined by (1.2), (1.5) and (1.6) satisfy the assumptions (A1) and (A2). It is well known that a travelling wave solution of (1.7) refers to a special translation invariant solution with the form $x_n(t) = \varphi(n - ct)$, $n \in \mathbb{Z}$, $t \in \mathbb{R}$ for a wave profile $\varphi(\cdot) : \mathbb{R} \to \mathbb{R}$ with an unknown wave speed $c \in \mathbb{R}$. Letting $\xi = n - ct$, it is easy to see that such a profile must satisfy the following equation:

$$-c\varphi'(\xi) = -\varphi(\xi) + \sum_{i=1}^{m} \alpha_i \int_0^\tau J_i(s) f(\varphi(\xi - i + cs))\,\mathrm{d}s + a \int_0^\tau J_{m+1}(s) f(\varphi(\xi + cs))\,\mathrm{d}s$$
$$+ \sum_{j=1}^{\ell} \beta_j \int_0^\tau J_{m+1+j}(s) f(\varphi(\xi + j + cs))\,\mathrm{d}s. \tag{1.8}$$

If $\varphi(\cdot)$ is strictly monotone, then we say φ is a *travelling wave front*. Note that the study of travelling wave solutions for partial differential equations and lattice dynamical systems has drawn considerable attention in past decades (see, e.g., [3, 9, 18, 19, 23, 25, 30, 42, 46, 49]).

In order to state our results later, we first recall the main results of theorems 1.1–1.4 of [48] as follows.

Proposition 1.1. *Assume* (A1)–(A3)*, then the following results hold.*

(1) *There exists a* $c_1^* \leqslant 0$ *such that for each* $c_1 \leqslant c_1^*$*, system* (1.7) *has a non-decreasing leftward travelling wave solution* $\phi_{c_1}(n - c_1 t)$ *which satisfies*

$$\phi_{c_1}(-\infty) = 0 \qquad \text{and} \qquad \phi_{c_1}(+\infty) = K.$$

Moreover, for any $c_1 < c_1^*$, $\phi_{c_1}(\xi) > 0$ *for all* $\xi \in \mathbb{R}$,

$$\lim_{\xi \to -\infty} \phi_{c_1}(\xi) \mathrm{e}^{-\lambda_1(c_1)\xi} = 1 \qquad \text{and} \qquad \lim_{\xi \to -\infty} \phi_{c_1}'(\xi) \mathrm{e}^{-\lambda_1(c_1)\xi} = \lambda_1(c_1),$$

where $\lambda_1(c)$ *is the smallest positive root of characteristic function (see* (2.2)*) of* (1.8) *at* 0 *and for* $c < 0$.

(2) *There exists a* $c_2^* \geqslant 0$ *such that for each* $c_2 \geqslant c_2^*$, *system (1.7) has a non-increasing rightward travelling wave solution* $\psi_{c_2}(n - c_2 t)$ *which satisfies*

$$\psi_{c_2}(-\infty) = K \qquad \text{and} \qquad \psi_{c_2}(+\infty) = 0.$$

Moreover, for any $c_2 > c_2^*$, $\psi_{c_2}(\xi) > 0$ *for all* $\xi \in \mathbb{R}$,

$$\lim_{\xi\to+\infty} \psi_{c_2}(\xi)\mathrm{e}^{-\lambda_3(c_2)\xi} = 1 \qquad \text{and} \qquad \lim_{\xi\to+\infty} \psi'_{c_2}(\xi)\mathrm{e}^{-\lambda_3(c_2)\xi} = \lambda_3(c_2),$$

where $\lambda_3(c)$ *is the largest negative root of characteristic function (see (2.2)) of (1.8) at* 0 *and for* $c > 0$.

The problems of travelling wave solutions are important in the study of various evolution equations, which provide significant applications in biology, chemistry, epidemiology and physics, see [11, 19, 21, 22, 36, 37, 44]. On the other hand, another important topic in those equations is the interactions of travelling wave solutions, which is crucially related to the pattern formation problem, see [7, 8, 24, 31] for more details. Mathematically, this phenomenon can be described by the so-called *entire solutions* that are defined in the whole space and for all time $t \in \mathbb{R}$ (see definition 1.2). Moreover, the entire solution can help us with the mathematical understanding of transient dynamics and the structures of the global attractor [32]. Recently, there have been quite a few works devoted to the interactions of travelling fronts and the entire solutions, see e.g., [1, 2, 10, 12, 13, 15, 16, 28, 32, 38, 47] for reaction–diffusion equations with and without delays, [39, 40] for delayed lattice differential equations with global interaction, and [14, 33, 41, 45] for some reaction–diffusion model systems. For other related results on entire solutions, we refer the reader to [26, 27, 35] and the references cited therein.

Based on the existence of travelling wave front solutions of (1.7), the purpose of this paper is to consider the interactions of travelling fronts for system (1.7) and establish some entire solutions to describe the phenomenon. Combining the leftward and rightward travelling fronts with different speeds and a spatially independent solution $\Gamma(\cdot)$ (solution of (2.1)), some new types of entire solutions are established. More precisely, inspired by the work of Hamel and Nadirashvili [15], we can construct appropriate subsolutions and derive some upper estimates. Then we show the existence of entire solutions by using the comparison principle. Before stating our main result, we first give the following definition.

Definition 1.2.

(1) *A sequence of functions* $\Phi(t) := \{\phi_n(t)\}_{n\in\mathbb{Z}}$, $t \in \mathbb{R}$, *is called an entire solution of (1.7) if for any* $n \in \mathbb{Z}$, $\phi_n(t)$ *is differential for all* $t \in \mathbb{R}$ *and* $\Phi(t)$ *satisfies (1.7) for all* $n \in \mathbb{Z}$ *and* $t \in \mathbb{R}$.

(2) *Let* $m \in \mathbb{N}$ *and* $p, p_0 \in \mathbb{R}^m$. *We say that a sequence of functions* $\Phi_p(t) := \{\Phi_{n,p}(t)\}_{n\in\mathbb{Z}}$ *converges to a function* $\Phi_{p_0}(t) := \{\Phi_{n,p_0}(t)\}_{n\in\mathbb{Z}}$ *in the sense of topology* $\mathcal{T}$ *if, for any compact set* $S \subset \mathbb{Z}\times\mathbb{R}$, *the functions* $\Phi_{n,p}(t)$ *and* $\Phi'_{n,p}(t)$ *converge uniformly in* S *to* $\Phi_{n,p_0}(t)$ *and* $\Phi'_{n,p_0}(t)$, *respectively, as* p *tends to* p_0.

(3) *Let* $\phi_{c_1}(n - c_1 t)$ *and* $\psi_{c_2}(n - c_2 t)$ *be the leftward and rightward travelling wave solutions of (1.7) (as decided in proposition 1.1) which have wave speeds* $c_1 < c_1^*$ *and* $c_2 > c_2^*$, *respectively. Then we denote*

$$A_{c_1} := \inf\{A > 0 |\ \phi_{c_1}(z)\mathrm{e}^{-\lambda_1(c_1)z} \leqslant A \text{ for all } z \in \mathbb{R}\},$$
$$B_{c_2} := \inf\{B > 0 |\ \phi_{c_2}(z)\mathrm{e}^{-\lambda_3(c_2)z} \leqslant B \text{ for all } z \in \mathbb{R}\}.$$

According to definition 1.2, we know that travelling wave solutions of (1.7) are special examples of the entire solutions. Throughout this paper, we always assume that conditions (A1)–(A3) hold. For convenience, let $\Gamma(\cdot)$ be the spatially independent solution of (1.7) connecting 0 and K, i.e. let the solution of (2.1) (see lemma 2.2) and $\lambda^* > 0$ be the unique root of the characteristic equation of (2.1) at the trivial equilibrium (see lemma 2.1). The main existence result is stated as follows.

Theorem 1.3. *For any $h_1, h_2, h_3 \in \mathbb{R}$, $c_1 < c_1^*$, $c_2 > c_2^*$ and $\chi_1, \chi_2, \chi_3 \in \{0, 1\}$ with $\chi_1 + \chi_2 + \chi_3 \geqslant 2$, there exists an entire solution $\Phi_p(t) = \{\Phi_{n,p}(t)\}_{n\in\mathbb{Z}}$ of (1.7) such that*

$$\max\left\{\chi_1\phi_{c_1}(n - c_1t + h_1), \chi_2\psi_{c_2}(n - c_2t + h_2), \chi_3\Gamma(t + h_3)\right\} \leqslant \Phi_{n,p}(t) \leqslant \min\left\{K, \Pi_1(n, t), \Pi_2(n, t), \Pi_3(n, t)\right\} \tag{1.9}$$

for $(n, t) \in \mathbb{Z} \times \mathbb{R}$, where $p := p_{\chi_1,\chi_2,\chi_3} = (\chi_1c_1, \chi_2c_2, \chi_1h_1, \chi_2h_2, \chi_3h_3)$, and

$$\Pi_1(n, t) := \chi_1\phi_{c_1}(n - c_1t + h_1) + \chi_2B_{c_2}e^{\lambda_3(c_2)(n-c_2t+h_2)} + \chi_3e^{\lambda^*(t+h_3)}, \tag{1.10}$$

$$\Pi_2(n, t) := \chi_1A_{c_1}e^{\lambda_1(c_1)(n-c_1t+h_1)} + \chi_2\psi_{c_2}(n - c_2t + h_2) + \chi_3e^{\lambda^*(t+h_3)}, \tag{1.11}$$

$$\Pi_3(n, t) := \chi_1A_{c_1}e^{\lambda_1(c_1)(n-c_1t+h_1)} + \chi_2B_{c_2}e^{\lambda_3(c_2)(n-c_2t+h_2)} + \chi_3\Gamma(t + h_3). \tag{1.12}$$

Moreover, various qualitative features of the entire solutions are also investigated in section 4.

The rest of the paper is organized as follows. In section 2, we first investigate the existence and asymptotic behaviour of spatially independent solutions Γ of (1.7). Some existence and comparison theorems for solutions, supersolutions and subsolutions of (1.7) are also established. According to the preliminaries derived in section 2, we prove the existence result of theorem 1.3 in section 3. Some qualitative properties of the entire solutions are further investigated in section 4.

2. Preliminaries

In this section, we first investigate the existence and asymptotic behaviour of spatially independent solutions of (1.7). Then we prove the well-posedness of initial value problem of (1.7), and establish some comparison theorems for supersolutions and subsolutions of (1.7).

First, we consider the spatially independent solutions of (1.7), that is, solutions of the following delay differential equation:

$$x'(t) = -x(t) + \int_0^\tau J(s)f(x(t - s))\,ds, \tag{2.1}$$

where $J(s)$ is defined by

$$J(s) := \sum_{i=1}^{m}\alpha_iJ_i(s) + aJ_{m+1}(s) + \sum_{j=1}^{\ell}\beta_jJ_{m+1+j}(s), \qquad s \in [0, \tau].$$

Obviously, the characteristic functions for (2.1) and (1.8) with respect to the trivial equilibrium can be represented by

$$\Delta_1(\lambda) := f'(0)\int_0^\tau J(s)e^{-\lambda s}\,ds - \lambda - 1,$$

$$\Delta_2(\lambda, c) := -f'(0)\int_0^\tau\left[\sum_{i=1}^{m}\alpha_iJ_i(s)e^{-\lambda i} + aJ_{m+1}(s) + \sum_{j=1}^{\ell}\beta_jJ_{m+1+j}(s)e^{\lambda j}\right]e^{\lambda cs}\,ds - c\lambda + 1 \tag{2.2}$$

respectively, for $\lambda \in \mathbb{R}$ and $c \in \mathbb{R}$. Then we have the following relation for the roots of $\Delta_1(\lambda)$ and $\Delta_2(\lambda, c)$.

Lemma 2.1. *The equation* $\Delta_1(\lambda) = 0$ *has a unique root* $\lambda^* > 0$. *Furthermore, if* $m = \ell$, $\alpha_i = \beta_i$ *and* $J_i(\cdot) = J_{m+1+i}(\cdot)$, $i = 1, \ldots, m$, *then*

$$-c_1\lambda_1(c_1) > \lambda^* \quad \text{and} \quad -c_2\lambda_3(c_2) > \lambda^* \quad \text{for any } c_1 < c_1^* \text{ and } c_2 > c_2^*.$$

Proof. Since $\Delta_1(0) = f'(0)(a + \alpha + \beta) - 1 > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\Delta_1(\lambda) = -f'(0)\int_0^\tau sJ(s)\mathrm{e}^{-\lambda s}\,\mathrm{d}s - 1 < 0$$

and $\lim_{\lambda\to+\infty}\Delta_1(\lambda) = -\infty$, it is easy to see that the equation $\Delta_1(\lambda) = 0$ has a unique root $\lambda^* > 0$.

Now we prove the second assertion of this lemma. By our assumptions, we know that

$$J(s) = 2\sum_{i=1}^m \alpha_i J_i(s) + aJ_{m+1}(s).$$

Suppose our assertion is false, then there exists a $c_1 < c_1^*$ such that $-c_1\lambda_1(c_1) \leqslant \lambda^*$ or a $c_2 > c_2^*$ such that $-c_2\lambda_3(c_2) \leqslant \lambda^*$. We consider the first case. Since $\Delta_2(\lambda_1(c_1), c_1) = 0$, we have

$$\begin{aligned}
0 \geqslant\ & -c_1\lambda_1(c_1) - \lambda^* \\
=\ & f'(0)\int_0^\tau \Bigg[\sum_{i=1}^m \alpha_i J_i(s)\big(\mathrm{e}^{\lambda_1(c_1)i} + \mathrm{e}^{-\lambda_1(c_1)i}\big) + aJ_{m+1}(s)\Bigg]\mathrm{e}^{\lambda_1(c_1)c_1 s}\,\mathrm{d}s - 1 - \lambda^* \\
>\ & f'(0)\int_0^\tau \Bigg[2\sum_{i=1}^m \alpha_i J_i(s) + aJ_{m+1}(s)\Bigg]\mathrm{e}^{-\lambda^* s}\,\mathrm{d}s - 1 - \lambda^* \\
=\ & f'(0)\int_0^\tau J(s)\mathrm{e}^{-\lambda^* s}\,\mathrm{d}s - 1 - \lambda^* = 0.
\end{aligned}$$

This contradiction shows that $-c_1\lambda_1(c_1) > \lambda^*$ for any $c_1 < c_1^*$. Similarly, we can show that $-c_2\lambda_3(c_2) > \lambda^*$ for any $c_2 > c_2^*$. This completes the proof. □

Next, we consider the existence and asymptotic behaviour for solutions of (2.1).

Lemma 2.2. *There exists a solution* $\Gamma(t) : \mathbb{R} \to \mathbb{R}$ *of equation (2.1) such that*

$$\Gamma'(t) \geqslant 0, \qquad \Gamma(t) > 0, \qquad \Gamma(t) \leqslant \mathrm{e}^{\lambda^* t} \qquad \text{for all } t \in \mathbb{R}$$

and satisfying

$$\Gamma(+\infty) = K \qquad \text{and} \qquad \lim_{t\to-\infty}\Gamma(t)\mathrm{e}^{-\lambda^* t} = 1.$$

Moreover, if $f \in C^1([0, \infty), [0, \infty))$, *then* $\Gamma'(t) > 0$ *for all* $t \in \mathbb{R}$.

Proof. The proof is similar to that of theorem 2.1 of [42] which uses the technique of monotone iteration scheme. Here we only sketch the outline.

Let $C(\mathbb{R}, \mathbb{R})$ be the space of continuous real functions on $\mathbb{R}$. We also define an operator $T : C(\mathbb{R}, [0, K]) \to C(\mathbb{R}, \mathbb{R})$ by

$$T(\phi)(t) = \int_{-\infty}^t \mathrm{e}^{-(t-s)}\left(\int_0^\tau J(r)f(\phi(s-r))\,\mathrm{d}r\right)\mathrm{d}s.$$

Then the rest of the proof is divided into the following three steps.

Step 1. It is easy to see that the following results hold:

(i) $T : C(\mathbb{R}, [0, K]) \to C(\mathbb{R}, [0, K])$;

(ii) $T(\phi)(t) \geqslant T(\psi)(t)$ for $\phi, \psi \in C(\mathbb{R}, [0, K])$ with $\phi(t) \geqslant \psi(t)$;
(iii) $T(\phi)(t)$ is increasing in $\mathbb{R}$ for $\phi \in C(\mathbb{R}, [0, K])$ with $\phi(t)$ is increasing in $\mathbb{R}$.

Step 2. For any fixed $\varepsilon \in (0, 1)$ and sufficiently large $q > 1$, we define

$$\overline{\phi}(t) = \min\left\{K, \mathrm{e}^{\lambda^* t}\right\} \quad \text{and} \quad \underline{\phi}(t) = \max\left\{0, \left(1 - q\mathrm{e}^{\varepsilon\lambda^* t}\right)\mathrm{e}^{\lambda^* t}\right\} \qquad \text{for all } t \in \mathbb{R}.$$

Then, by direct computations, we obtain

$$0 \leqslant \underline{\phi}(t) \leqslant \overline{\phi}(t) \leqslant K, \quad T(\overline{\phi})(t) \leqslant \overline{\phi}(t) \quad \text{and} \quad T(\underline{\phi})(t) \geqslant \underline{\phi}(t) \quad \text{for all } t \in \mathbb{R}.$$

Step 3. Using the monotone iteration technique, we can show that equation (2.1) admits a solution $\Gamma(t)$ which satisfies

$$\Gamma'(t) \geqslant 0 \quad \text{and} \quad \underline{\phi}(t) \leqslant \Gamma(t) \leqslant \overline{\phi}(t) \qquad \text{for all } \ t \in \mathbb{R}.$$

Thus,

$$\lim_{t\to-\infty} \Gamma(t)\mathrm{e}^{-\lambda^* t} = 1, \quad \Gamma(+\infty) \in (0, K] \quad \text{and} \quad 0 < \Gamma(t) \leqslant \mathrm{e}^{\lambda^* t} \qquad \text{for all } t \in \mathbb{R}.$$

Moreover, one can easily verify that $\Gamma(+\infty) = K$.

If $f \in C^1([0, \infty), [0, \infty))$, then $\Gamma(t) \in C^2(\mathbb{R})$ and for all $t \in \mathbb{R}$,

$$\Gamma''(t) = -\Gamma'(t) + \int_0^\tau J(s) f'(\Gamma(t-s))\Gamma'(t-s)\,\mathrm{d}s \geqslant -\Gamma'(t).$$

Suppose that there exists a $t_1 \in \mathbb{R}$ such that $\Gamma'(t_1) = 0$. Then, $\Gamma'(t_1) \geqslant \Gamma'(t)\mathrm{e}^{t-t_1}$ for all $t < t_1$ which implies that $\Gamma'(t) = 0$ for all $t \leqslant t_1$. Hence $\Gamma(t_1) = \lim_{t\to-\infty} \Gamma(t) = 0$ which contradicts to $\Gamma(t_1) > 0$. Therefore, $\Gamma'(t) > 0$ for all $t \in \mathbb{R}$. The proof is complete. □

Now we consider the existence problem for the initial value problem of (1.7) with the initial condition:

$$x_n(s) = \varphi_n(s), \qquad n \in \mathbb{Z}, \qquad s \in [r - \tau, r], \tag{2.3}$$

where $r \in \mathbb{R}$ is an any given constant. We also establish some comparison theorems for supersolution and subsolutions of (1.7). The definitions of supersolution and subsolution are given as follows.

Definition 2.3. *A sequence of continuous differential functions $\{x_n(t)\}_{n\in\mathbb{Z}}$, $t \in [r, b)$, $b > r$, is called a supersolution (or a subsolution) of (1.7) on $[r, b)$ if for all $n \in \mathbb{Z}$ and $t \in [r, b)$,*

$$x_n'(t) \geqslant (or \leqslant) \sum_{i=1}^m \alpha_i \int_0^\tau J_i(s) f(x_{n-i}(t-s))\,\mathrm{d}s + a\int_0^\tau J_{m+1}(s) f(x_n(t-s))\,\mathrm{d}s$$
$$+ \sum_{j=1}^{\ell} \beta_j \int_0^\tau J_{m+1+j}(s) f(x_{n+j}(t-s))\,\mathrm{d}s - x_n(t). \tag{2.4}$$

By definition 2.3, we have the following results.

Lemma 2.4. *We consider the problem of (1.7) and (2.3).*

(1) *For any $\varphi = \{\varphi_n\}_{n\in\mathbb{Z}}$ with $\varphi_n \in C\big([r-\tau, r], [0, K]\big)$, (1.7) admits a unique solution $x(t;\varphi) = \big\{x_n(t;\varphi)\big\}_{n\in\mathbb{Z}}$ on $[r, +\infty)$ such that $x_n(s) = \varphi_n(s)$ and $0 \leqslant x_n(t) \leqslant K$ for $n \in \mathbb{Z}$, $s \in [r-\tau, r]$ and $t \in [r-\tau, +\infty)$.*
(2) *Suppose $\big\{x_n^+(t)\big\}_{n\in\mathbb{Z}}$ and $\big\{x_n^-(t)\big\}_{n\in\mathbb{Z}}$ are a supersolution and subsolution of (1.7) on $[r, +\infty)$, respectively, such that $0 \leqslant x_n^-(t), x_n^+(t) \leqslant K$ and $x_n^+(s) \geqslant x_n^-(s)$ for $n \in \mathbb{Z}$, $s \in [r-\tau, r]$ and $t \in [r-\tau, +\infty)$, then $x_n^+(t) \geqslant x_n^-(t)$ for $n \in \mathbb{Z}$, $t \geqslant r$.*

Proof.

(1) We denote

$$H_n[x](t) := \sum_{i=1}^{m} \alpha_i \int_0^{\tau} J_i(s) f(x_{n-i}(t-s))\, \mathrm{d}s + a \int_0^{\tau} J_{m+1}(s) f(x_n(t-s))\, \mathrm{d}s$$
$$+ \sum_{j=1}^{\ell} \beta_j \int_0^{\tau} J_{m+1+j}(s) f(x_{n+j}(t-s))\, \mathrm{d}s, \tag{2.5}$$

$S := \Big\{ x(t) = \{x_n(t)\}_{n\in\mathbb{Z}} \Big| x_n(\cdot) \in C\big([r-\tau, +\infty), [0, K]\big) \text{ and satisfies (2.3)} \Big\}$,

and define an operator $F = \{F_n\}_{n\in\mathbb{Z}} : S \to S$ by

$$F_n[x](t) := \begin{cases} \varphi_n(r)\mathrm{e}^{-(t-r)} + \displaystyle\int_r^t H_n[x](s)\mathrm{e}^{-(t-s)}\, \mathrm{d}s, & \text{for } n \in \mathbb{Z}, t > r, \\ \varphi_n(t), & \text{for } n \in \mathbb{Z}, t \in [r-\tau, r]. \end{cases}$$

For any $\lambda > 0$, we set

$$X_\lambda := \Big\{ x(t) = \{x_n(t)\}_{n\in\mathbb{Z}} \Big| x_n(\cdot) \in C\big([r-\tau, +\infty), \mathbb{R}\big), \sup_{n\in\mathbb{Z}, t\geqslant r-\tau} |x_n(t)|\mathrm{e}^{-\lambda t} < +\infty \Big\},$$

and

$$\|x\|_\lambda := \sup_{n\in\mathbb{Z}, t\geqslant r-\tau} |x_n(t)|\mathrm{e}^{-\lambda t}.$$

It is easy to see that $\big(X_\lambda, \|\cdot\|_\lambda\big)$ is a Banach space and $S \subset X_\lambda$ is a closed subset of X_λ. Moreover, we can choose a sufficiently large $\lambda > 0$ such that $F : S \to S$ is a contracting map. Hence, there exists a unique fixed point $x(\cdot) \in S$ of F which is a solution of (1.7) and (2.3) on $[r, +\infty)$.

(2) Put $w_n(t) := x_n^-(t) - x_n^+(t)$, $n \in \mathbb{Z}$, $t \geqslant r-\tau$, then $w_n(t)$ and $Z(t) := \sup_{n\in\mathbb{Z}}\{w_n(t)\}$ are continuous and bounded on $[r-\tau, +\infty)$. Let $\delta > 0$ be such that $\delta > f'(0)\int_0^\tau J(s)\, \mathrm{d}s$. Suppose the assertion of (2) is false, then there exists a $t_0 > r$ such that $Z(t_0) > 0$ and

$$Z(t_0)\mathrm{e}^{-\delta t_0} = \max_{t\geqslant r-\tau} Z(t)\mathrm{e}^{-\delta t} > Z(s)\mathrm{e}^{-\delta s}, \qquad \forall s \in [r-\tau, t_0). \tag{2.6}$$

It is easy to see that there exists a sequence $\{n_k\}_{k=1}^{+\infty}$ such that

$$w_{n_k}(t_0) > 0, \qquad \forall k \geqslant 1 \qquad \text{and} \qquad \lim_{k\to+\infty} w_{n_k}(t_0) = Z(t_0).$$

Let $\{t_k\}_{k=1}^{+\infty} \subset (r, t_0]$ be such that

$$w_{n_k}(t_k)\mathrm{e}^{-\delta t_k} = \max_{t\in[r,t_0]} w_{n_k}(t)\mathrm{e}^{-\delta t}. \tag{2.7}$$

It then follows from (2.6) that $\lim_{k\to+\infty} t_k = t_0$. Since

$$w_{n_k}(t_0)\mathrm{e}^{-\delta t_0} \leqslant w_{n_k}(t_k)\mathrm{e}^{-\delta t_k} \leqslant Z(t_k)\mathrm{e}^{-\delta t_k} \leqslant Z(t_0)\mathrm{e}^{-\delta t_0},$$

we obtain that $\lim_{k\to+\infty} w_{n_k}(t_k) = Z(t_0)$. In view of (2.7), for each $k \geqslant 1$, we have

$$0 \leqslant \mathrm{e}^{\delta t_k} \frac{\mathrm{d}}{\mathrm{d}t}\Big(w_{n_k}(t)\mathrm{e}^{-\delta t}\Big)\Big|_{t=t_k^-} = \frac{\mathrm{d}}{\mathrm{d}t} w_{n_k}(t_k) - \delta w_{n_k}(t_k)$$
$$= -(\delta+1)w_{n_k}(t_k) + \sum_{i=1}^{m} \alpha_i \int_0^\tau J_i(s)\left[f(x_{n_k-i}^-(t_k-s)) - f(x_{n_k-i}^+(t_k-s))\right] \mathrm{d}s$$
$$+ a\int_0^\tau J_{m+1}(s)\left[f(x_{n_k}^-(t_k-s)) - f(x_{n_k}^+(t_k-s))\right] \mathrm{d}s$$
$$+ \sum_{j=1}^{\ell} \beta_j \int_0^\tau J_{m+1+j}(s)\Big[f(x_{n_k+j}^-(t_k-s)) - f(x_{n_k+j}^+(t_k-s))\Big] \mathrm{d}s. \tag{2.8}$$

Since $|f(u) - f(v)| \leqslant f'(0)|u - v|$ for all $u, v \in [0, K]$, it follows from (2.8) that

$$0 \leqslant -(\delta+1)w_{n_k}(t_k) + f'(0)\sum_{i=1}^{m}\alpha_i\int_0^{\tau} J_i(s)\max\left\{0, Z(t_k - s)\right\}\mathrm{d}s$$
$$+ af'(0)\int_0^{\tau} J_{m+1}(s)\max\left\{0, Z(t_k - s)\right\}\mathrm{d}s$$
$$+ f'(0)\sum_{j=1}^{\ell}\beta_j\int_0^{\tau} J_{m+1+j}(s)\max\left\{0, Z(t_k - s)\right\}\mathrm{d}s.$$

Taking $k \to +\infty$, we obtain

$$0 \leqslant -(\delta+1)Z(t_0) + f'(0)\sum_{i=1}^{m}\alpha_i\int_0^{\tau} J_i(s)\mathrm{e}^{\delta(t_0-s)}\max\left\{0, Z(t_0 - s)\mathrm{e}^{-\delta(t_0-s)}\right\}\mathrm{d}s$$
$$+ af'(0)\int_0^{\tau} J_{m+1}(s)\mathrm{e}^{\delta(t_0-s)}\max\left\{0, Z(t_0 - s)\mathrm{e}^{-\delta(t_0-s)}\right\}\mathrm{d}s$$
$$+ f'(0)\sum_{j=1}^{\ell}\beta_j\int_0^{\tau} J_{m+1+j}(s)\mathrm{e}^{\delta(t_0-s)}\max\left\{0, Z(t_0 - s)\mathrm{e}^{-\delta(t_0-s)}\right\}\mathrm{d}s$$
$$\leqslant \left\{-\delta + f'(0)\int_0^{\tau}\left(\sum_{i=1}^{m}\alpha_i J_i(s) + aJ_{m+1}(s) + \sum_{j=1}^{\ell}\beta_j J_{m+1+j}(s)\right)\mathrm{e}^{-\delta s}\,\mathrm{d}s\right\}Z(t_0)$$
$$\leqslant \left(-\delta + f'(0)\int_0^{\tau} J(s)\,\mathrm{d}s\right)Z(t_0).$$

Therefore $Z(t_0) \leqslant 0$ and which contradicts to $Z(t_0) > 0$. Hence, $x_n^+(t) \geqslant x_n^-(t)$ for $n \in \mathbb{Z}$ and $t \geqslant r$. The proof is complete. □

Moreover, we give an *a priori* estimate of solutions of (1.7) in the following lemma.

Lemma 2.5. *Assume that $x(t;\varphi) = \left\{x_n(t;\varphi)\right\}_{n\in\mathbb{Z}}$ is a solution of (1.7) with the initial value $\varphi = \{\varphi_n\}_{n\in\mathbb{Z}}$ satisfying $\varphi_n \in C\big([r-\tau, r], [0, K]\big)$, then there exists a positive constant M, independent of φ and r, such that for any $n \in \mathbb{Z}$, $t > r + \tau$ and $h \geqslant 0$,*

$$\left|x_n'(t;\varphi)\right| \leqslant M \qquad \text{and} \qquad \left|x_n'(t+h;\varphi) - x_n'(t;\varphi)\right| \leqslant Mh. \tag{2.9}$$

Proof. For convenience, we denote $x_n(t;\varphi)$ by $x_n(t)$. From lemma 2.4, we know that

$$0 \leqslant x_n(t) \leqslant K \qquad \text{for } n \in \mathbb{Z} \quad \text{and} \quad t \in [r-\tau, +\infty).$$

Then it is easy to see that

$$\left|x_n'(t)\right| \leqslant M_1 := (a + \alpha + \beta)\max_{u\in[0,K]} f(u) + K = 2K$$

for $n \in \mathbb{Z}$ and $t \in [r, +\infty)$. Moreover, for $n \in \mathbb{Z}$ and $t > r + \tau$, we have

$$\left|x_n'(t+h) - x_n'(t)\right| \leqslant \left|x_n(t+h) - x_n(t)\right|$$
$$+ f'(0)\sum_{i=1}^{m}\alpha_i\int_0^{\tau} J_i(s)\left|x_{n-i}(t+h-s) - x_{n-i}(t-s)\right|\mathrm{d}s$$
$$+ af'(0)\int_0^{\tau} J_{m+1}(s)\left|x_n(t+h-s) - x_n(t-s)\right|\mathrm{d}s$$
$$+ f'(0)\sum_{j=1}^{\ell}\beta_j\int_0^{\tau} J_{m+1+j}(s)\left|x_{n+j}(t+h-s) - x_{n+j}(t-s)\right|\mathrm{d}s$$
$$\leqslant M_2 h := [f'(0)(a + \alpha + \beta) + 1]M_1 h.$$

Taking $M := \max\{M_1, M_2\}$, then the assertion of this lemma follows. □

Lemma 2.6. *Let* $x_n^+(t) \in C\big([r-\tau,+\infty),[0,+\infty)\big)$ and $x_n^-(t) \in C\big([r-\tau,+\infty),(-\infty,K]\big)$ *be such that* $x_n^+(s) \geqslant x_n^-(s)$ *for all* $n \in \mathbb{Z}$ *and* $s \in [r-\tau, r]$*, and*

$$\frac{\mathrm{d}}{\mathrm{d}t}x_n^+(t) \geqslant -x_n^+(t) + f'(0)\Big[\sum_{i=1}^{m}\alpha_i \int_0^\tau J_i(s)x_{n-i}^+(t-s)\,\mathrm{d}s + a\int_0^\tau J_{m+1}(s)x_n^+(t-s)\,\mathrm{d}s + \sum_{j=1}^{\ell}\beta_j \int_0^\tau J_{m+1+j}(s)x_{n+j}^+(t-s)\,\mathrm{d}s\Big],$$

$$\frac{\mathrm{d}}{\mathrm{d}t}x_n^-(t) \leqslant -x_n^-(t) + f'(0)\Big[\sum_{i=1}^{m}\alpha_i \int_0^\tau J_i(s)x_{n-i}^-(t-s)\,\mathrm{d}s + a\int_0^\tau J_{m+1}(s)x_n^-(t-s)\,\mathrm{d}s + \sum_{j=1}^{\ell}\beta_j \int_0^\tau J_{m+1+j}(s)x_{n+j}^-(t-s)\,\mathrm{d}s\Big]$$

for all $n \in \mathbb{Z}$ *and* $t > r$*. Then* $x_n^+(t) \geqslant x_n^-(t)$ *for all* $n \in \mathbb{Z}$ *and* $t \geqslant r$*.*

Proof. The proof is similar to part (2) of lemma 2.4. We omit it here. □

3. Existence of entire solutions

In this section, we will use the properties of previous sections to obtain an appropriate upper estimate for solutions of (1.7) and then prove the existence result of theorem 1.3.

For any $k \in \mathbb{Z}^+$, $h_1, h_2, h_3 \in \mathbb{R}$, $c_1 < c_1^*, c_2 > c_2^*$ and $\chi_1, \chi_2, \chi_3 \in \{0, 1\}$ with $\chi_1 + \chi_2 + \chi_3 \geqslant 2$, we denote

$$\varphi_n^k(s) := \max\big\{\chi_1\phi_{c_1}(n - c_1 s + h_1),\ \chi_2\psi_{c_2}(n - c_2 s + h_2),\ \chi_3\Gamma(s + h_3)\big\},$$
$$\underline{x}_n(t) := \max\big\{\chi_1\phi_{c_1}(n - c_1 t + h_1),\ \chi_2\psi_{c_2}(n - c_2 t + h_2),\ \chi_3\Gamma(t + h_3)\big\},$$

where $s \in [-k-\tau, -k]$ and $t > -k$. Let $x^k(t) = \{x_n^k(t)\}_{n\in\mathbb{Z}}$ be the unique solution of the following initial value problem

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}x_n^k(t) = \displaystyle\sum_{i=1}^{m}\alpha_i \int_0^\tau J_i(s)f\big(x_{n-i}^k(t-s)\big)\,\mathrm{d}s + a\int_0^\tau J_{m+1}(s)f\big(x_n^k(t-s)\big)\,\mathrm{d}s \\ \quad + \displaystyle\sum_{j=1}^{\ell}\beta_j \int_0^\tau J_{m+1+j}(s)f\big(x_{n+j}^k(t-s)\big)\,\mathrm{d}s - x_n^k(t), \qquad n \in \mathbb{Z}, \quad t > -k, \\ x_n^k(s) = \varphi_n^k(s), \qquad n \in \mathbb{Z}, \quad s \in [-k-\tau, -k]. \end{cases} \tag{3.1}$$

Then, by lemma 2.4, we have $\underline{x}_n(t) \leqslant x_n^k(t) \leqslant K$ for all $n \in \mathbb{Z}$ and $t \geqslant -k$. The following result provides the appropriate upper estimate of $x^k(t)$.

Lemma 3.1. *The unique solution* $x^k(t) = \{x_n^k(t)\}_{n\in\mathbb{Z}}$ *of (3.1) satisfies*

$$\underline{x}_n(t) \leqslant x_n^k(t) \leqslant \min\big\{K, \Pi_1(n,t), \Pi_2(n,t), \Pi_3(n,t)\big\}$$

for any $n \in \mathbb{Z}$ *and* $t \geqslant -k-\tau$*. Note that* $\Pi_1(n,t)$*,* $\Pi_2(n,t)$ *and* $\Pi_3(n,t)$ *are defined in* theorem 1.3.

Proof. We only prove $x_n^k(t) \leqslant \Pi_1(n,t)$ for all $n \in \mathbb{Z}$ and $t \geqslant -k-\tau$. The other cases can also been proved in the same way. Assume $\chi_1 = 1$ and set

$$Z_n^k(t) := x_n^k(t) - \phi_{c_1}(n - c_1 t + h_1).$$

By assumption (A2) and direct computation, we obtain

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} Z_n^k(t) \leqslant f'(0) \sum_{i=1}^{m} \alpha_i \int_0^\tau J_i(s) Z_{n-i}^k(t-s)\,\mathrm{d}s + af'(0) \int_0^\tau J_{m+1}(s) Z_n^k(t-s)\,\mathrm{d}s \\ \quad + f'(0) \sum_{j=1}^{\ell} \beta_j \int_0^\tau J_{m+1+j}(s) Z_{n+j}^k(t-s)\,\mathrm{d}s - Z_n^k(t), \\ Z_n^k(s) = \varphi_n^k(s) - \phi_{c_1}(n - c_1 s + h_1), \end{cases} \tag{3.2}$$

where $n \in \mathbb{Z}$, $t > -k$, $s \in [-k-\tau, -k]$. Taking

$$V_n(t) := \chi_2 B_{c_2} \mathrm{e}^{\lambda_3(c_2)(n - c_2 t + h_2)} + \chi_3 \mathrm{e}^{\lambda^*(t+h_3)},$$

it is easy to verify that

$$\frac{\mathrm{d}}{\mathrm{d}t} V_n(t) = f'(0) \sum_{i=1}^{m} \alpha_i \int_0^\tau J_i(s) V_{n-i}(t-s)\,\mathrm{d}s + af'(0) \int_0^\tau J_{m+1}(s) V_n(t-s)\,\mathrm{d}s$$
$$+ f'(0) \sum_{j=1}^{\ell} \beta_j \int_0^\tau J_{m+1+j}(s) V_{n+j}(t-s)\,\mathrm{d}s - V_n(t), \qquad \text{for } n \in \mathbb{Z},\ t > -k.$$

According to definition 1.2 and lemma 2.2, we have

$$\psi_{c_2}(z) \leqslant B_{c_2} \mathrm{e}^{\lambda_3(c_2) z} \quad \text{and} \quad \Gamma(z) \leqslant \mathrm{e}^{\lambda^* z} \qquad \text{for all } z \in \mathbb{R},$$

which implies

$$\begin{aligned} V_n(s) &:= \chi_2 B_{c_2} \mathrm{e}^{\lambda_3(c_2)(n - c_2 s + h_2)} + \chi_3 \mathrm{e}^{\lambda^*(s+h_3)} \geqslant \chi_2 \psi_{c_2}(n - c_2 s + h_2) + \chi_3 \Gamma(s + h_3) \\ &\geqslant \varphi_n^k(s) - \phi_{c_1}(n - c_1 s + h_1) \\ &= Z_n^k(s) \qquad \text{for } s \in [-k-\tau, -k]. \end{aligned}$$

It then follows from lemma 2.6 that

$$Z_n^k(t) \leqslant V_n(t) \qquad \text{for all } n \in \mathbb{Z} \quad \text{and} \quad t > -k - \tau,$$

that is,

$$x_n^k(t) \leqslant \phi_{c_1}(n - c_1 t + h_1) + \chi_2 B_{c_2} \mathrm{e}^{\lambda_3(c_2)(n - c_2 t + h_2)} + \chi_3 \mathrm{e}^{\lambda^*(t+h_3)} = \Pi_1(n, t).$$

If $\chi_1 = 0$, then the assertion $x_n^k(t) \leqslant \Pi_1(n, t)$ obviously reduces to

$$x_n^k(t) \leqslant \chi_2 B_{c_2} \mathrm{e}^{\lambda_3(c_2)(n - c_2 t + h_2)} + \chi_3 \mathrm{e}^{\lambda^*(t+h_3)}.$$

Hence the assertion of the lemma follows. The proof is complete. □

Now we prove the result of theorem 1.3.

Proof of theorem 1.3. By lemmas 2.4 and 3.1, we have

$$\underline{x}_n(t) \leqslant x_n^k(t) \leqslant x_n^{k+1}(t) \leqslant \min\left\{K, \Pi_1(n,t), \Pi_2(n,t), \Pi_3(n,t)\right\}$$

for any $n \in \mathbb{Z}$ and $t \geqslant -k-\tau$. Using the *a priori* estimate of lemma 2.5 and the diagonal extraction process, there exists a subsequence $x^{k_l}(t) = \{x^{k_l}(t)\}_{l\in\mathbb{N}}$ of $x^k(t)$ such that $x^{k_l}(t)$ converges to a function $\Phi_p(t) = \left\{\Phi_{n,p}(t)\right\}_{n\in\mathbb{Z}}$ in the sense of topology $\mathcal{T}$. Since $x_n^k(t) \leqslant x_n^{k+1}(t)$ for any $t > -k$, we have

$$\lim_{k\to+\infty} x_n^k(t) = \Phi_{n,p}(t) \qquad \text{for any} \quad (n,t) \in \mathbb{Z} \times \mathbb{R}.$$

The limit function is unique, whence all of the functions $x^k(t)$ converge to the function $\Phi_p(t)$ in the sense of topology $\mathcal{T}$ as $k \to +\infty$. Clearly, $\Phi_p(t)$ is an entire solution of (1.7) satisfying (1.9). The proof is complete.

4. Qualitative properties of the entire solutions

In addition to the existence result of theorem 1.3, in this section we further investigate some qualitative properties of the entire solutions.

For any $N \in \mathbb{Z}$ and $\gamma \in \mathbb{R}$, we denote the regions $T^i_{N,\gamma}$, $i = 1, \ldots, 6$ by

$$T^1_{N,\gamma} := [N, \infty) \times [\gamma, \infty), \quad T^2_{N,\gamma} := (-\infty, N] \times [\gamma, \infty), \quad T^3_{N,\gamma} := \mathbb{Z} \times [\gamma, \infty),$$
$$T^4_{N,\gamma} := (-\infty, N] \times (-\infty, \gamma], \quad T^5_{N,\gamma} := [N, \infty) \times (-\infty, \gamma], \quad T^6_{N,\gamma} := \mathbb{Z} \times (-\infty, \gamma].$$

Various qualitative properties of the entire solutions are stated in the following.

Proposition 4.1. *Let $\Phi_p(t) = \{\Phi_{n,p}(t)\}_{n\in\mathbb{Z}}$ be the entire solution of (1.7) as stated in* theorem 1.3, *then the following properties hold.*

(1) *$\Phi_{n,p}(t) > 0$ and $\Phi'_{n,p}(t) \geqslant 0$ for all $n \in \mathbb{Z}$ and $t \in \mathbb{R}$. Moreover, if $f \in C^1([0,\infty),[0,\infty))$ then $\Phi'_{n,p}(t) > 0$ for all $n \in \mathbb{Z}$ and $t \in \mathbb{R}$.*
(2) *$\lim_{t\to+\infty} \sup_{n\in\mathbb{Z}} \left|\Phi_{n,p}(t) - K\right| = 0$ and $\lim_{t\to-\infty} \sup_{|n|\leqslant N} \Phi_{n,p}(t) = 0$ for any $N \in \mathbb{N}$.*
(3) *If $\chi_1 = 1$ then $\lim_{n\to+\infty} \sup_{t\geqslant T} \left|\Phi_{n,p}(t) - K\right| = 0$ for any $T \in \mathbb{R}$.*
(4) *If $\chi_2 = 1$ then $\lim_{n\to-\infty} \sup_{t\geqslant T} \left|\Phi_{n,p}(t) - K\right| = 0$ for any $T \in \mathbb{R}$.*
(5) *If $\chi_3 = 1$, $m = \ell$, $\alpha_i = \beta_i$ and $J_i(\cdot) = J_{m+1+i}(\cdot)$ for $i = 1, \ldots, m$, then*
$$\Phi_{n,p}(t) \sim \Gamma(t + h_3) \sim \mathrm{e}^{\lambda^*(t+h_3)} \qquad \text{as} \quad t \to -\infty \quad \text{for every} \quad n \in \mathbb{Z}.$$
(6) *If $\chi_3 = 0$ then for any $n \in \mathbb{Z}$, there exist constants $D_n > C_n > 0$ such that*
$$C_n \mathrm{e}^{\vartheta(c_1,c_2)t} \leqslant \Phi_{n,p}(t) \leqslant D_n \mathrm{e}^{\vartheta(c_1,c_2)t}$$
for $t \ll -1$, here $\vartheta(c_1, c_2) := \min\{-c_1\lambda_1(c_1), -c_2\lambda_3(c_2)\}$.
(7) *For any $n \in \mathbb{Z}$, $\Phi_{n,p}(t)$ is decreasing with respect to h_2 and increasing with respect to h_1 and h_3, respectively.*
(8) *For any $N \in \mathbb{Z}$ and $\gamma \in \mathbb{R}$, $\Phi_p(t)$ converges to K in the sense of topology $\mathcal{T}$ as $h_i \to +\infty$ and uniformly on $(n,t) \in T^i_{N,\gamma}$ for $i = 1, 3$. If $h_2 \to -\infty$ then $\Phi_p(t)$ converges to K in the sense of topology $\mathcal{T}$ and uniformly on $(n,t) \in T^2_{N,\gamma}$.*

Proof. The assertions for parts (2)–(4) and (6)–(8) are direct consequences of (1.9). Therefore, we only prove the results of parts (1) and (5).

(1) From (1.9), one can see that $\Phi_{n,p}(t) > 0$ for all $n \in \mathbb{Z}$ and $t \in \mathbb{R}$. Since

$$x^k_n(t) \geqslant \underline{x}_n(t) \geqslant \underline{x}_n(s) = \varphi^n(s) \qquad \text{for all } (n,t) \in \mathbb{Z} \times [-k, +\infty)$$
$$\text{and} \quad s \in [-k-\tau, -k],$$

by lemma 2.4, we have $\frac{\mathrm{d}}{\mathrm{d}t} x^k_n(t) \geqslant 0$ for $(n,t) \in \mathbb{Z} \times (-k, +\infty)$, which yields to $\Phi'_{n,p}(t) \geqslant 0$ for all $n \in \mathbb{Z}$ and $t \in \mathbb{R}$.

Moreover, if $f \in C^1([0,\infty),[0,\infty))$, then

$$\begin{aligned}\Phi''_{n,p}(t) = & -\Phi'_{n,p}(t) + \sum_{i=1}^{m} \alpha_i \int_0^\tau J_i(s) f'(\Phi_{n-i,p}(t-s))\Phi'_{n-i,p}(t-s)\,\mathrm{d}s \\ & + a\int_0^\tau J_{m+1}(s) f'(\Phi_{n,p}(t-s))\Phi'_{n,p}(t-s)\,\mathrm{d}s \\ & + \sum_{j=1}^{\ell} \beta_j \int_0^\tau J_{m+1+j}(s) f'(\Phi_{n+j,p}(t-s))\Phi'_{n+j,p}(t-s)\,\mathrm{d}s, \end{aligned} \tag{4.1}$$

where $n \in \mathbb{Z}$ and $t \in \mathbb{R}$. Hence, for any $r < t$, we have

$$\Phi'_{n,p}(t) = \Phi'_{n,p}(r)\mathrm{e}^{-(t-r)} + \int_r^t h(s)\mathrm{e}^{-(t-s)}\,\mathrm{d}s, \tag{4.2}$$

where

$$\begin{aligned} h(t) = &\sum_{i=1}^{m}\alpha_i \int_0^{\tau} J_i(s) f'(\Phi_{n-i,p}(t-s))\Phi'_{n-i,p}(t-s)\,\mathrm{d}s \\ &+ a\int_0^{\tau} J_{m+1}(s) f'(\Phi_{n,p}(t-s))\Phi'_{n,p}(t-s)\,\mathrm{d}s \\ &+ \sum_{j=1}^{\ell}\beta_j \int_0^{\tau} J_{m+1+j}(s) f'(\Phi_{n+j,p}(t-s))\Phi'_{n+j,p}(t-s)\,\mathrm{d}s. \end{aligned}$$

Clearly, $h(t) \geqslant 0$ for all $n \in \mathbb{Z}$ and $t \in \mathbb{R}$. Suppose on the contrary that there exists a $(n_0, t_0) \in \mathbb{Z} \times \mathbb{R}$ such that $\Phi'_{n_0,p}(t_0) = 0$, then it follows from (4.2) that $\Phi'_{n_0,p}(r) = 0$ for all $r \leqslant t_0$. Hence $\Phi_{n_0,p}(t) = \Phi_{n_0,p}(t_0)$ for all $t \leqslant t_0$, which implies that $\lim_{t\to-\infty} \Phi_{n_0,p}(t) = \Phi_{n_0,p}(t_0)$. However, following from (1.9), we have $\lim_{t\to-\infty} \Phi_{n_0,p}(t) = 0$ and $\Phi_{n_0,p}(t_0) > 0$. This contradiction implies that $\Phi'_{n,p}(t) > 0$ for all $t \in \mathbb{R}$.

(5) By lemma 2.1, we know that

$$\min\big\{-c_1\lambda_1(c_1), -c_2\lambda_3(c_2)\big\} > \lambda^* \qquad \text{for any } c_1 < c_1^* \quad \text{and} \quad c_2 > c_2^*.$$

Then (1.9) implies

$$\begin{aligned} \Gamma(t+h_3) \leqslant \Phi_{n,p}(t) &\leqslant \chi_1 A_{c_1}\mathrm{e}^{\lambda_1(c_1)(n-c_1t+h_1)} + \chi_2 B_{c_2}\mathrm{e}^{\lambda_3(c_2)(n-c_2t+h_2)} + \Gamma(t+h_3) \\ &\leqslant \chi_1 A_{c_1}\mathrm{e}^{\lambda_1(c_1)(n-c_1t+h_1)} + \chi_2 B_{c_2}\mathrm{e}^{\lambda_3(c_2)(n-c_2t+h_2)} + \mathrm{e}^{\lambda^*(t+h_3)}. \end{aligned}$$

Since $\lim_{t\to-\infty} \Gamma(t)e^{-\lambda^* t} = 1$, the statement of (5) holds obviously. The proof is complete. □

Moreover, according to the assumption $\chi_1, \chi_2, \chi_3 \in \{0, 1\}$ with $\chi_1 + \chi_2 + \chi_3 \geqslant 2$ in theorem 1.3, we further denote the entire solution $\Phi_p(t)$ of (1.7) by

$$\Phi_p(t) := \begin{cases} \Phi_{p_0}(t) = \{\Phi_{n,p_0}(t)\}_{n\in\mathbb{Z}}, & \text{if } (\chi_1, \chi_2, \chi_3) = (1, 1, 1); \\ \Phi_{p_1}(t) = \{\Phi_{n,p_1}(t)\}_{n\in\mathbb{Z}}, & \text{if } (\chi_1, \chi_2, \chi_3) = (0, 1, 1); \\ \Phi_{p_2}(t) = \{\Phi_{n,p_2}(t)\}_{n\in\mathbb{Z}}, & \text{if } (\chi_1, \chi_2, \chi_3) = (1, 0, 1); \\ \Phi_{p_3}(t) = \{\Phi_{n,p_3}(t)\}_{n\in\mathbb{Z}}, & \text{if } (\chi_1, \chi_2, \chi_3) = (1, 1, 0), \end{cases} \tag{4.3}$$

where $p = p_{\chi_1,\chi_2,\chi_3} = (\chi_1c_1, \chi_2c_2, \chi_1h_1, \chi_2h_2, \chi_3h_3)$, $p_0 = (c_1, c_2, h_1, h_2, h_3)$, $p_1 = (0, c_2, 0, h_2, h_3)$, $p_2 = (c_1, 0, h_1, 0, h_3)$ and $p_3 = (c_1, c_2, h_1, h_2, 0)$. Then we have the following convergence results.

Proposition 4.2. *From (4.3), we have the following properties.*

(1) *For any $N \in \mathbb{Z}$ and $\gamma \in \mathbb{R}$, $\Phi_{p_0}(t)$ converges (in the sense of topology $\mathcal{T}$) to*

$$\begin{cases} \Phi_{p_1}(t) \text{ as } h_1 \to -\infty, \text{ and uniformly on } (n,t) \in T^4_{N,\gamma}; \\ \Phi_{p_2}(t) \text{ as } h_2 \to +\infty, \text{ and uniformly on } (n,t) \in T^5_{N,\gamma}; \\ \Phi_{p_3}(t) \text{ as } h_3 \to -\infty, \text{ and uniformly on } (n,t) \in T^6_{N,\gamma}. \end{cases}$$

(2) *For any $N \in \mathbb{Z}$ and $\gamma \in \mathbb{R}$, $\Phi_{p_1}(t)$ converges (in the sense of topology $\mathcal{T}$) to*

$$\begin{cases} \Gamma(t+h_3) \textit{ as } h_2 \to +\infty, \text{ and uniformly on } (n,t) \in T^5_{N,\gamma}; \\ \psi_{c_2}(n-c_2t+h_2) \textit{ as } h_3 \to -\infty, \text{ and uniformly on } (n,t) \in T^6_{N,\gamma}. \end{cases}$$

(3) *For any* $N \in \mathbb{Z}$ *and* $\gamma \in \mathbb{R}$, $\Phi_{p_2}(t)$ *converges (in the sense of topology* $\mathcal{T}$*) to*

$$\begin{cases} \Gamma(t+h_3) \text{ as } h_1 \to -\infty, \text{ and uniformly on } (n,t) \in T^4_{N,\gamma}; \\ \phi_{c_1}(n-c_1t+h_1) \text{ as } h_3 \to -\infty, \text{ and uniformly on } (n,t) \in T^6_{N,\gamma}. \end{cases}$$

(4) *For any* $N \in \mathbb{Z}$ *and* $\gamma \in \mathbb{R}$, $\Phi_{p_3}(t)$ *converges (in the sense of topology* $\mathcal{T}$*) to*

$$\begin{cases} \psi_{c_2}(n-c_2t+h_2) \text{ as } h_1 \to -\infty, \text{ and uniformly on } (n,t) \in T^4_{N,\gamma}, \\ \phi_{c_1}(n-c_1t+h_1) \text{ as } h_2 \to +\infty, \text{ and uniformly on } (n,t) \in T^5_{N,\gamma}. \end{cases}$$

(5) *For any* $h_1, h_2, h_1^*, h_2^* \in \mathbb{R}$, *there exists* $(n_0, t_0) \in \mathbb{Z} \times \mathbb{R}$, *depending on* c_1, c_2, h_1, h_2, h_1^*, h_2^*, *such that*

$$\Phi_{n,p_3}(t) = \Phi_{n+n_0,p_3^*}(t+t_0) \qquad \text{for all} \quad (n,t) \in \mathbb{Z} \times \mathbb{R}$$

if and only if

$$\frac{c_2(h_1-h_1^*) - c_1(h_2-h_2^*)}{c_2-c_1} \in \mathbb{Z}. \tag{4.4}$$

Here $p_3^* := (c_1, c_2, h_1^*, h_2^*, 0)$.

Proof.

(1) We only prove the case that $\Phi_{p_0}(t)$ converges to $\Phi_{p_3}(t)$ in the sense of topology $\mathcal{T}$ as $h_3 \to -\infty$, and uniformly on $(n,t) \in T^6_{N,\gamma}$. The proofs for the other cases are similar.

For $(\chi_1, \chi_2, \chi_3) = (1,1,1)$, we denote $\varphi^k(s) = \{\varphi^k_n(s)\}_{n\in\mathbb{Z}}$ by $\varphi^k_{p_0}(s) = \{\varphi^k_{n,p_0}(s)\}_{n\in\mathbb{Z}}$ and $x^k(t) = \{x^k_n(t)\}_{n\in\mathbb{Z}}$ by $x^k_{p_0}(t) = \{x^k_{n,p_0}(t)\}_{n\in\mathbb{Z}}$, respectively. Similarly, when $(\chi_1, \chi_2, \chi_3) = (1,1,0)$, we denote $\varphi^k(s)$ by $\varphi^k_{p_3}(s)$ and $x^k(t)$ by $x^k_{p_3}(t)$, respectively. Let $W^k(t) = \{W^k_n(t)\}_{n\in\mathbb{Z}} := x^k_{p_0}(t) - x^k_{p_3}(t)$, then $0 \leqslant W^k_n(t) \leqslant K$ for all $(n,t) \in \mathbb{Z} \times (-k, +\infty)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} W^k_n(t) \leqslant f'(0)\Big[\sum_{i=1}^m \alpha_i \int_0^\tau J_i(s) W^k_{n-i}(t-s)\,\mathrm{d}s + a\int_0^\tau J_{m+1}(s) W^k_n(t-s)\,\mathrm{d}s + \sum_{j=1}^\ell \beta_j \int_0^\tau J_{m+1+j}(s) W^k_{n+j}(t-s)\,\mathrm{d}s\Big] - W^k_n(t)$$

for $n \in \mathbb{Z}$ and $t > -k$. Note that

$$W^k_n(s) = \varphi^k_{n,p_0}(s) - \varphi^k_{n,p_3}(s) \leqslant \Gamma(s+h_3) \leqslant \mathrm{e}^{\lambda^*(s+h_3)} \qquad \text{for} \quad s \in [-k-\tau, -k]$$

and the function $\widehat{W}(t) = \mathrm{e}^{\lambda^*(t+h_3)}$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} \widehat{W}(t) = f'(0)\Big[\sum_{i=1}^m \alpha_i \int_0^\tau J_i(s) \widehat{W}(t-s)\,\mathrm{d}s + a\int_0^\tau J_{m+1}(s) \widehat{W}(t-s)\,\mathrm{d}s + \sum_{j=1}^\ell \beta_j \int_0^\tau J_{m+1+j}(s) \widehat{W}(t-s)\,\mathrm{d}s\Big] - \widehat{W}(t) \qquad \text{for } t > -k.$$

It then follows from lemma 2.6 that

$$0 \leqslant W^k_n(t) \leqslant \mathrm{e}^{\lambda^*(t+h_3)} \qquad \text{for all } (n,t) \in \mathbb{Z} \times [-k, +\infty).$$

Since $\lim_{k\to+\infty} x^k_{p_0}(t) = \Phi_{p_0}(t)$ and $\lim_{k\to+\infty} x^k_{p_3}(t) = \Phi_{p_3}(t)$, we obtain

$$0 \leqslant \Phi_{n,p_0}(t) - \Phi_{n,p_3}(t) \leqslant \mathrm{e}^{\lambda^*(t+h_3)} \qquad \text{for all } (n,t) \in \mathbb{Z} \times \mathbb{R},$$

which implies that $\Phi_{p_0}(t)$ converges to $\Phi_{p_3}(t)$ as $h_3 \to -\infty$ uniformly on $(n,t) \in T^6_{N,\gamma}$ for any $\gamma \in \mathbb{R}$. For any sequence h_3^ℓ with $h_3^\ell \to -\infty$ as $\ell \to +\infty$, the functions $\Phi_{p^\ell}(t)$

(here $p^\ell := (c_1, c_2, h_1, h_2, h_3^\ell)$) converge to a solution of (1.7) (up to extraction of some subsequence) in the sense of topology $\mathcal{T}$, which turns out to be $\Phi_{p_3}(t)$. The limit does not depend on the sequence h_3^ℓ, whence all of the functions $\Phi_{p_0}(t)$ converge to $\Phi_{p_3}(t)$ in the sense of topology $\mathcal{T}$ as $h_3 \to -\infty$. Hence the assertion of this part follows.

The proofs of parts (2)–(4) are similar to that of part (1), and omitted.

(5) When $\chi_1 = \chi_2 = 1$ and $\chi_3 = 0$, by (1.9), we have, for any $n \geqslant 0$ and $t \in \mathbb{R}$,

$$0 \leqslant \Phi_{n,p_3}(t) - \phi_{c_1}(n - c_1 t + h_1) \leqslant B_{c_2} e^{\lambda_3(c_2)(n-c_2t+h_2)} \leqslant B_{c_2} e^{\lambda_3(c_2)(-c_2t+h_2)}$$

which implies that

$$\lim_{t\to-\infty} \sup_{n\geqslant 0} \left|\Phi_{n,p_3}(t) - \phi_{c_1}(n - c_1 t + h_1)\right| = 0. \tag{4.5}$$

Similarly, we obtain

$$\lim_{t\to-\infty} \sup_{n\leqslant 0} \left|\Phi_{n,p_3}(t) - \psi_{c_2}(n - c_2 t + h_2)\right| = 0. \tag{4.6}$$

For any $h_1, h_2, h_1^*, h_2^* \in \mathbb{R}$, suppose that there exists a $(n_0, t_0) \in \mathbb{Z} \times \mathbb{R}$ such that $\Phi_{n,p_3}(t) = \Phi_{n+n_0,p_3^*}(t + t_0)$ for all $(n, t) \in \mathbb{Z} \times \mathbb{R}$. Then, from (4.5), we obtain

$$\lim_{t\to-\infty} \sup_{n\geqslant 0} \left|\Phi_{n+n_0,p_3^*}(t + t_0) - \phi_{c_1}(n - c_1 t + h_1)\right| = 0$$

and

$$\lim_{t\to-\infty} \sup_{n\geqslant -n_0} \left|\Phi_{n+n_0,p_3^*}(t + t_0) - \phi_{c_1}((n + n_0) - c_1(t + t_0) + h_1^*)\right| = 0.$$

Hence,

$$\lim_{t\to-\infty} \sup_{n\geqslant \max\{0,-n_0\}} \left|\phi_{c_1}((n + n_0) - c_1(t + t_0) + h_1^*) - \phi_{c_1}(n - c_1 t + h_1)\right| = 0. \tag{4.7}$$

Let $\{t_n\}_{n\in\mathbb{N}}$ be such that $n - c_1 t_n = 0$ for all $n \in \mathbb{N}$, then (4.7) implies

$$n_0 - c_1 t_0 + h_1^* = h_1 \tag{4.8}$$

as $n \to +\infty$. Similarly, by (4.6), we obtain

$$n_0 - c_2 t_0 + h_2^* = h_2. \tag{4.9}$$

Solving (4.8) and (4.9), we obtain

$$n_0 = \frac{c_2(h_1 - h_1^*) - c_1(h_2 - h_2^*)}{c_2 - c_1} \quad \text{and} \quad t_0 = \frac{(h_1 - h_1^*) - (h_2 - h_2^*)}{c_2 - c_1}. \tag{4.10}$$

Hence condition (4.4) holds obviously.

Conversely, if the condition (4.4) hold, one can easily verify that $\Phi_{n,p_3}(t) = \Phi_{n+n_0,p_3^*}(t+t_0)$ for all $(n, t) \in \mathbb{Z} \times \mathbb{R}$, where (n_0, t_0) is given by (4.10). This completes the proof. □

Remark 4.3. If the function $f(\cdot)$ is odd, then the function $\Psi_p(t) := -\Phi_p(t)$ is also an entire solution of (1.7) which satisfies the similar properties of $\Phi_p(t)$ as stated in propositions 4.1–4.2.

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