TRAVELING FRONTS AND ENTIRE SOLUTIONS IN
PARTIALLY DEGENERATE REACTION-DIFFUSION SYSTEMS
WITH MONOSTABLE NONLINEARITY

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Abstract. This paper is concerned with traveling fronts and entire solutions
for a class of monostable partially degenerate reaction-diffusion systems. It is
known that the system admits traveling wave solutions. In this paper, we first
prove the monotonicity and uniqueness of the traveling wave solutions, and
the existence of spatially independent solutions. Combining traveling fronts
with different speeds and a spatially independent solution, the existence and
various qualitative features of entire solutions are then established by using
comparison principle. As applications, we consider a reaction-diffusion model
with a quiescent stage in population dynamics and a man-environment-man
epidemic model in physiology.

1. Introduction. In the past decades, quite a few reaction-diffusion systems that
some but not all diffusion coefficients are zeroes called partially degenerate reaction-
diffusion systems, have been introduced to give an accurate description of a wide
variety of phenomena in population biology, epidemiology, and so on. See the model
\[
\begin{align*}
  u_t(x,t) &= d u_{xx}(x,t) - a_{11} u(x,t) + a_{12} v(x,t), \\
  v_t(x,t) &= -a_{22} v(x,t) + g(u(x,t)),
\end{align*}
\]
which is proposed by Capasso and Maddalena [1] to study the fecally-orally trans-
mitted diseases in the European Mediterranean regions, and the following reaction-
diffusion system in [11]
\[
\begin{align*}
  u_t(x,t) &= d u_{xx}(x,t) + f_1(u(x,t)) - \gamma_1 u(x,t) + \gamma_2 v(x,t), \\
  v_t(x,t) &= \gamma_1 u(x,t) - \gamma_2 v(x,t),
\end{align*}
\]
which describes a species population where the individuals alternate between mobile
and stationary states, and only the mobile reproduce. For more details, we refer
to [6,14,30,31] and the references cited therein.

Traveling wave solutions of partially degenerate reaction-diffusion systems, espe-
cially for the models (1) and (2) have been widely studied. For example, Xu and
Zhao [28] proved the existence, uniqueness and stability of traveling fronts of (1)
with bistable nonlinearity, and Zhao and Wang [32] established the existence of a
minimal wave speed of (1) with monostable nonlinearity. For system (2), Zhang and

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Zhao [30] considered the asymptotic behavior of solutions, and Zhang and Li [31] further established the monotonicity and uniqueness of traveling wave solutions. Recently, Fang and Zhao [6] studied the traveling fronts and spreading speed of a general partially degenerate reaction-diffusion system. Li [15] further considered the traveling fronts for a class of partially degenerate reaction-diffusion systems that can have three or more equilibria. However, for the monostable case, these studies only considered the existence and non-existence of the traveling wave solutions. The first purpose of this paper is to further consider the asymptotic behavior, monotonicity and uniqueness of the monostable traveling wave solutions.

Although the study of traveling wave solutions is an important issue of reaction-diffusion equations, it is not enough for mathematical understanding of the dynamical structure of solutions. In fact, traveling wave solutions are only special examples of the so-called entire solutions that are defined for all time $t \in \mathbb{R}$ and for all point $x \in \mathbb{R}$. Recently, there are quite a few works devoted to the entire solutions of scalar reaction-diffusion equations with and without delays, see e.g., [4, 5, 7–9, 12, 13, 16, 19, 24–26, 29]. On the other hand, to the best of our knowledge, there were only four papers studying on the entire solutions of reaction-diffusion systems [10, 20, 22, 27], where the existence of entire solutions for some specific reaction-diffusion model systems was established by employing comparison principle. The second purpose of this paper is to extend the works [10, 20, 22, 27] to a large class of monostable partially degenerate reaction-diffusion systems.

More precisely, in this paper, we consider the traveling fronts and entire solutions of the following partially degenerate reaction-diffusion system

$$
\begin{align*}
&u_t(x, t) = du_{xx}(x, t) + f(u(x, t), v(x, t)), \\
v_t(x, t) = -\beta v(x, t) + g(u(x, t)),
\end{align*}
$$

which is a generalized version of the models (1) and (2). The nonlinearity of (3) is induced by the functions $f$ and $g$, which satisfies the following conditions:

(A1): $g \in C^2([0, K_1], \mathbb{R})$, $g(0) = 0$, $K_2 = g(K_1)/\beta > 0$, $f \in C^2([0, K_1] \times [0, K_2], \mathbb{R})$, $f(0, 0) = f(K_1, K_2) = 0$ and $f(u, \frac{1}{\beta}g(u)) > 0$ for $u \in (0, K_1)$, where $K_1$ is a positive constant;

(A2): $g'(u) > 0$ for $u \in [0, K_1]$, $\beta \partial_1 f(K_1, K_2) + g'(K_1) \partial_2 f(K_1, K_2) < 0$, and $\partial_2 f(u, v) > 0$ for $(u, v) \in [0, K_1] \times (0, K_2]$;

(A3): $g(u) \leq g'(0)u$ for $u \in [0, K_1]$ and $f(u, v) \leq \partial_1 f(0, 0)u + \partial_2 f(0, 0)v$ for $(u, v) \in [0, K_1] \times [0, K_2]$.

We shall use a similar argument as in [10, 20, 22] to consider the entire solutions of (3). The idea is to study the solutions $w^n(x, t) = (u^n(x, t), v^n(x, t))$ of Cauchy problems for (3) starting at times $-n (n \in \mathbb{N})$ with appropriate initial conditions. By constructing appropriate sub- and supersolutions, some new entire solutions are obtained by passing the limit $n \to \infty$. Although our method is similar to the works [10, 20, 22], the technique details are different. For example, for the partially degenerate system (3), the sequence functions $v^n(x, t)$ are not smooth enough with respect to $x$ due to zero diffusion coefficient in $v$-equation, and hence its convergence is not ensured. To obtain a convergent subsequence, we have to make $\{v^n(x, t)\}$ possess a property which is similar to a global Lipschitz condition with respect to $x$ (Lemma 15).

According to (A1), system (3) has two equilibria $0 := (0, 0)$ and $K := (K_1, K_2)$. Conditions (A1) and (A3) imply that $\beta \partial_1 f(0, 0)g'(0)\partial_2 f(0, 0) \geq \frac{2\beta}{K_1} f'(\frac{K_1}{2}) \cdot \frac{1}{\beta} g'(\frac{K_1}{2}) > 0$. Hereafter, a solution $w(x, t) := (u(x, t), v(x, t))$ of system (3) is called a
traveling wave solution connecting $0$ and $K$ with speed $c$, if $(u(x,t),v(x,t)) = (\phi_c(x),\psi_c(x))$, $\xi := x + ct$ for some function $(\phi_c,\psi_c) : \mathbb{R} \to [0,K] := [0,K_1] \times [0,K_2]$ which satisfies
\[
\begin{align*}
\phi''_c(\xi) &= d\phi''_c(\xi) + f(\phi_c(\xi),\psi_c(\xi)), \\
\psi''_c(\xi) &= -\beta \psi_c(\xi) + g(\phi_c(\xi)),
\end{align*}
\] (4)
and
\[
(\phi_c(-\infty),\psi_c(-\infty)) = (0,0), \ (\phi_c(+\infty),\psi_c(+\infty)) = (K_1,K_2).
\] (5)
We say $(\phi_c,\psi_c)$ is a traveling (wave) front if $(\phi_c(\cdot),\psi_c(\cdot))$ is monotone. Under the assumptions $(A_1)$-$(A_3)$, it is easy to prove that there exists a number $c_{\min} > 0$ (defined precisely in Lemma 1) such that system (3) has a traveling front for any $c \geq c_{\min}$ and has no traveling wave solutions for $c \in (0,c_{\min})$, see e.g., Fang and Zhao [6].

Throughout this paper, we always use the usual notations for the standard ordering in $\mathbb{R}^2$. We also use $\| \cdot \|$ to denote the Euclidean norm in $\mathbb{R}^2$.

Now, we state our main results as follows.

**Theorem 1.** Assume $(A_1)$-$(A_3)$. Then, the following result holds:

(i) any traveling wave solution $\Phi_c(\cdot) = (\phi_c(\cdot),\psi_c(\cdot))$ of (3) with speed $c \geq c_{\min}$ satisfies $\phi'_c(\xi) > 0$ and $\psi'_c(\xi) > 0$ for all $\xi \in \mathbb{R}$;

(ii) for each $c \geq c_{\min}$, the traveling wave solutions of (3) with speed $c$ are unique up to translations.

**Theorem 2.** Assume $(A_1)$-$(A_3)$. Then, system (3) has a spatially independent solution $\Gamma(t) = (\Gamma_1(t),\Gamma_2(t))$ which satisfies
\[
\Gamma(+\infty) = K, \ \Gamma'(t) \geq 0, \ \lim_{t \to -\infty} \Gamma(t)e^{-\lambda^* t} = (1,b_*) \text{ and } \Gamma(t) \leq (1,b_*)e^{\lambda^* t}
\] for all $t \in \mathbb{R}$, where $\lambda^*$ is defined in Lemma 11 and $b_* = g'(0)/(\lambda^* + \beta)$.

In the sequel, we always assume $\Phi_c(\cdot) = (\phi_c(\cdot),\psi_c(\cdot))$ is a traveling wave solution of (3) with speed $c \geq c_{\min}$. To obtain the existence and qualitative properties of entire solutions other than the traveling fronts and the spatially independent solution, we need a stronger condition $(A_3)'$ as follows:

\[ (A_3)': g''(u) \leq g'(u) \text{ for } u \in [0,K_1], \ \partial_1 f(u,v) \leq \partial_1 f(0,0) \text{ and } \partial_2 f(u,v) \leq \partial_2 f(0,0) \text{ for } (u,v) \in [0,K]. \]

**Theorem 3.** Assume $(A_1)$, $(A_2)$ and $(A_3)'$. Then, for any $\theta_1,\theta_2 \in \mathbb{R}$ and $c_1,c_2 \geq c_{\min}$, there exists an entire solution $W_{c_1,c_2,\theta_1,\theta_2}(x,t) = (u(x,t),v(x,t)) : \mathbb{R}^2 \to (0,K_1] \times (0,K_2]$ of (3) such that
\[
\begin{align*}
\lim_{t \to -\infty} \sup_{x \geq 0} \| W_{c_1,c_2,\theta_1,\theta_2}(x,t) - \Phi_{c_1}(x+c_1 t + \theta_1) \| &= 0, \\
\lim_{t \to -\infty} \sup_{x \leq 0} \| W_{c_1,c_2,\theta_1,\theta_2}(x,t) - \Phi_{c_2}(-x+c_2 t + \theta_2) \| &= 0.
\end{align*}
\] (6) (7)

Furthermore, the following statements hold:

(i): $\frac{D}{D_t} u(x,t) > 0$ and $\frac{D}{D_t} v(x,t) > 0$ for any $(x,t) \in \mathbb{R}^2$.

(ii): $\lim_{t \to -\infty} \sup_{x \in \mathbb{R}} \| W_{c_1,c_2,\theta_1,\theta_2}(x,t) - K \| = 0$, and $\lim_{t \to -\infty} \sup_{x \in [x_1,x_2]} \| W_{c_1,c_2,\theta_1,\theta_2}(x,t) - K \| = 0$ for any $x_1 < x_2$.

(iii): For any $t_0 \in \mathbb{R}$, $\lim_{|x| \to +\infty} \sup_{t \in [t_0,\infty)} \| W_{c_1,c_2,\theta_1,\theta_2}(x,t) - K \| = 0$. 

(iv): For any \((x,t) \in \mathbb{R}^2\), \(W_{c_1,c_2,\theta_1,\theta_2}(x,t)\) is increasing with respect to \(\theta_i\), \(i = 1,2\).

(v): \(W_{c_1,c_2,\theta_1,\theta_2}(x,t) \to K\) as \(\theta_i \to +\infty\) uniformly on \((x,t) \in T^2_{N,a}\) for any \(N,a \in \mathbb{R}\), \(i = 1,2\), where \(T^2_{N,a} = [N, +\infty) \times [a, +\infty)\) and \(T^2_N = (-\infty, N] \times (-\infty, a]\).

(vi): If \(c_2 > c_{\min}\), then \(W_{c_1,c_2,\theta_1,\theta_2}(x,t) \to \Phi_{c_1}(x+c_1 t + \theta_1)\) as \(\theta_2 \to -\infty\) uniformly on \((x,t) \in [N, +\infty) \times (-\infty, a]\) for any \(N,a \in \mathbb{R}\), and if \(c_1 > c_{\min}\), then \(W_{c_1,c_2,\theta_1,\theta_2}(x,t) \to \Phi_{c_2}(-x+c_2 t + \theta_2)\) as \(\theta_1 \to -\infty\) uniformly on \((x,t) \in (-\infty, N] \times (-\infty, a]\) for any \(N,a \in \mathbb{R}\).

(vii): For each \(x \in \mathbb{R}\), there exist \(A_1(x) \gg 0\) and \(A_2(x) \gg 0\) such that for all \(t \ll -1\),

\[
A_1(x)e^{c_{\max} \lambda_1(c_{\max})t} \leq W_{c_1,c_2,\theta_1,\theta_2}(x,t) \leq A_2(x)e^{c_{\max} \lambda_1(c_{\max})t},
\]

where \(c_{\max} = \max\{c_1, c_2\}\) and \(\lambda_1(c)\) is defined in Lemma 1.

(viii): For any \((c_1^*, c_2^*) \neq (c_1, c_2)\) with \(c_1^*, c_2^* \geq c_{\min}\), there is no \((x_0, t_0) \in \mathbb{R}^2\) such that \(W_{c_1^*,c_2^*,\theta_1,\theta_2}(\cdot, x_0, t_0) \in \mathbb{R}^2\).

(ix): For any \(\theta_1, \theta_2 \in \mathbb{R}\), there exist \(x_0 = x_0(\theta_1, \theta_2, \theta_1^*, \theta_2^*)\) and \(t_0 = t_0(\theta_1, \theta_2, \theta_1^*, \theta_2^*)\) such that \(W_{c_1,c_2,\theta_1,\theta_2}(\cdot, x_0, t_0) \in \mathbb{R}^2\).

Let \(\Gamma(t)\) be the spatially independent solution of (3) given in Theorem 2. We can consider any combination of traveling fronts and the spatially independent solution to construct some entire solutions. For convenience, we define

\[
\max\{w_1, w_2\} = \{\max\{u_1, u_2\}, \max\{v_1, v_2\}\}
\]

and

\[
\min\{w_1, w_2\} = \{\min\{u_1, u_2\}, \min\{v_1, v_2\}\}
\]

for \(w_1 = (u_1, v_1)\) and \(w_2 = (u_2, v_2)\).

**Theorem 4.** Assume \((A_1), (A_2)\) and \((A_3)\). Then, for any \(\theta_1, \theta_2, \theta_3 \in \mathbb{R}, c_1, c_2 \geq c_{\min}\) and \(\chi_1, \chi_2 \in [0,1]\) with \(\chi_1 + \chi_2 \geq 1\), there exists an entire solution \(W(x,t) : \mathbb{R}^2 \to (0, K_1) \times (0, K_2)\) of (3) such that

\[
\max\{\chi_1 \Phi_{c_1}(x+c_1 t + \theta_1), \chi_2 \Phi_{c_2}(-x+c_2 t + \theta_2), \Gamma(t+\theta_3)\}
\]

\[
\leq W(x,t) \leq \min\{\chi_1 \Phi_{c_1}(x+p_1(t)) + \chi_2 \Phi_{c_2}(-x+p_2(t)) + (1-b_0)e^{\lambda(t+\theta_3)}, K\}
\]

for \((x,t) \in \mathbb{R} \times (-\infty, 0]\), where \(p_i(t) = p(t) = p(t) - c_i t - \theta_i \leq R(t)e^{\nu t}\) are monotone increasing functions on \((-\infty, T]\), \(T < 0, R > 0, \nu > 0\) are constants. Moreover, the assertions (i)-(iii) in Theorem 3 still hold for \(W(\cdot, \cdot)\) as for \(W_{c_1,c_2,\theta_1,\theta_2}(\cdot, \cdot)\) and there further hold:

(a): For each \(x \in \mathbb{R}\), \(W(x,t) \sim \Gamma(t+\theta_3) \sim (1-b_0)e^{\lambda(t+\theta_3)}\) as \(t \to -\infty\).

(b): If \(\chi_1 = 0\) (respectively, \(\chi_2 = 0\), then for any \(t \in \mathbb{R}\), the function \(W(x,t)\) is decreasing in \(x\) (respectively, increasing in \(x\)).

(c): For any \((x,t) \in \mathbb{R}^2\), \(W(x,t)\) is increasing with respect to \(\theta_i\), \(i = 1,2,3\).

(d): \(W(x,t) \to K\) as \(\theta_i \to +\infty\) uniformly on \((x,t) \in T^2_{N,a}\) for any \(N,a \in \mathbb{R}, i = 1,2,3\), where \(T^2_{N,a}\) and \(T^2_N\) are defined as (v) in Theorem 1 and \(T^2_N = \mathbb{R} \times [a, +\infty)\).

(e): Denote \(W(x,t)\) by \(W_{c_1,c_2,\theta_1,\theta_2,\theta_3}(x,t)\) when \(\chi_1 = \chi_2 = 1\) (Similarly, denote \(W_{c_1,\theta_1,\theta_2}(x,t)\) and \(W_{c_2,\theta_1,\theta_2}(x,t)\)).

(e1) If \(c_1 > c_{\min}\), then \(W_{c_1,c_2,\theta_1,\theta_2,\theta_3}(x,t) \to W_{c_2,\theta_1,\theta_2,\theta_3}(x,t)\) as \(\theta_1 \to -\infty\) uniformly on \((x,t) \in (-\infty, N] \times (-\infty, a]\) for any \(N,a \in \mathbb{R}\).

(e2) If \(c_2 > c_{\min}\), then \(W_{c_1,c_2,\theta_1,\theta_2,\theta_3}(x,t) \to W_{c_1,\theta_1,\theta_2,\theta_3}(x,t)\) as \(\theta_2 \to -\infty\) uniformly on \((x,t) \in (-\infty, N] \times (-\infty, a]\) for any \(N,a \in \mathbb{R}\).
uniformly on \((x,t) \in [N, +\infty) \times (-\infty, a]\) for any \(N, a \in \mathbb{R}\); and
\begin{align*}
&\text{(c3) } W_{c_1, c_2, \theta_1, \theta_2, \theta_3}(x,t) \rightarrow W_{c_1, c_2, \theta_1, \theta_2}(x,t) \text{ as } \theta_3 \rightarrow -\infty \text{ uniformly on } (x,t) \in \mathbb{R} \times (-\infty, a] \\
&\text{for any } a \in \mathbb{R}.
\end{align*}

(1): \(W_{c_2, \theta_2, \theta_3}(x,t) \rightarrow W_{c_2, \theta_2}(x,t) \text{ as } \theta_3 \rightarrow -\infty \text{ uniformly on } \mathbb{R} \times (-\infty, a]
\text{for any } a \in \mathbb{R}, \text{ and if } c_2 > c_{\min}, \text{ then } W_{c_2, \theta_2, \theta_3}(x,t) \rightarrow \Gamma(t + \theta_3) \text{ as } \theta_2 \rightarrow -\infty
\text{uniformly on } (x,t) \in [N, +\infty) \times (-\infty, a] \text{ for any } N, a \in \mathbb{R}. \text{ Similar results hold for } W_{c_1, \theta_1, \theta_3}(x,t).

Since \(\lambda^* < c\lambda_1(c)\) for any \(c \geq c_{\min}\) (Lemma 11), entire solutions \(W(x,t)\) established in Theorem 4 and \(W_{c_1, c_2, \theta_1, \theta_2}(x,t)\) in Theorem 3 have different decay rates when \(t \rightarrow -\infty\), and hence they are completely different.

**Remark 1.** When the solutions of (3) are assumed to range in \([0, K]\), system (3) can be decoupled by solving the second equation and transformed into the scalar equation with **infinite delay**:

\[
u(t) = -\beta \int_0^\infty u(x,t) e^{-\beta s} g(u(x,t-s))ds + f(u(x,t)) \quad (8)
\]

In [16, 23, 24], Li et al. considered the traveling fronts and entire solutions of a nonlocal reaction-diffusion equation with **finite delay** of the form:

\[
u(t) = -\beta \int_0^\infty u(x,t) e^{-\beta s} g(u(x,t-s))ds + f(u(x,t)) \quad (9)
\]

where \(\tau \geq 0\) is a finite constant. For the case \(c > c_{\min}\), the monotonicity and uniqueness of the traveling wave solutions of (3) can be obtained from the arguments of Wang et al. [23]. Thus, the new result in Theorem 1 is to guarantee such monotonicity and uniqueness when \(c = c_{\min}\).

As far as the entire solution is concerned, the argument of [16, 24] is similar to those of [4, 9, 12]. More precisely, they studied the solutions \(u^n(x,t)(n \in \mathbb{N})\) of Cauchy problems for (9) starting at times \(-n\) with appropriate initial conditions. By constructing appropriate subsolutions and supersolutions and establishing some priori estimates of solutions, the entire solutions are obtained by passing the limit \(n \rightarrow \infty\). However, for the equation with infinite delay, such as (8), a lack of regularizing effect occurs, see e.g., [21]. Thus, the sequence functions \(\{u^n(x,t)\}\) are not smooth enough, and hence its convergence is not ensured. Therefore, the existence and qualitative properties of the entire solutions of (8) and (3) (Theorems 3 and 4) can not be obtained directly from the results of [16, 24].

The rest of this paper is organized as follows. In Section 2, we first transform the corresponding wave system into a scalar problem with an integral term. This property is effectively used to investigate the traveling wave solutions of (3). Then, the asymptotic behavior of the wave profiles at \(\pm \infty\) is established by using the Ikehara’s theorem [2]. At the end of Section 2, we prove the monotonicity and uniqueness of the traveling wave solutions by using a sliding method, see e.g., Chen and Guo [3]. Section 3 is devoted to the existence and asymptotic behavior of the spatially independent solution. In Section 4, Theorems 3 and 4 are proved by using comparison principle and constructing appropriate sub-super-solution pairs. The method is inspired by Guo and Morita [9] and Chen and Guo [4], see also Wang et al., [16, 24]. As applications, the main results are applied to the models (1) and (2) in Section 5.
2. Properties of traveling wave solutions. In this section, we study the asymptotic behavior, monotonicity and uniqueness of traveling wave solutions of \((3)\). Throughout this section, we assume \((A_1)-(A_3)\).

First, we transform the wave system of \((3)\) into a scalar differential equation with an integral term. Let \(\Phi(\cdot) = (\phi(\cdot), \psi(\cdot))\) be a traveling wave solution of \((3)\) with speed \(c \geq c_{\text{min}}\). By the second equation of \((4)\) and \(\psi(-\infty) = 0\), we obtain
\[
\psi(\xi) = \frac{1}{c} \int_{-\infty}^{\xi} e^{-\frac{c}{2}(\xi-s)} g(\phi(s)) ds. \tag{10}
\]
Substituting \((10)\) into the first equation of \((4)\), we get
\[
c \phi'(\xi) = d \phi''(\xi) + f \left( \phi(\xi), \frac{1}{c} \int_{-\infty}^{\xi} e^{-\frac{c}{2}(\xi-s)} g(\phi(s)) ds \right). \tag{11}
\]
Conversely, if \(\phi(\xi)\) is a solution of \((11)\) with \(\phi(\cdot) \in [0, K_1]\), \(\phi(-\infty) = 0\) and \(\phi(+\infty) = K_1\), and let \(\psi(\xi)\) be defined by \((10)\), then \((\phi(\xi), \psi(\xi))\) is a traveling wave solution of \((3)\).

Consequently, we only need to consider the corresponding solutions \(\phi\) of \((11)\) with
\[
\phi(\cdot) \in [0, K_1], \ \phi(-\infty) = 0 \text{ and } \phi(+\infty) = K_1. \tag{12}
\]

For \(c > 0\) and \(\lambda \in \mathbb{C}\setminus\{-\frac{\beta}{c}\}\), define two functions as follows:
\[
\begin{align*}
\Delta_1(c, \lambda) &= d \lambda^2 - c \lambda + \partial_1 f(0,0) + \partial_2 f(0,0)g'(0)/(c\lambda + \beta), \\
\Delta_2(c, \lambda) &= d \lambda^2 - c \lambda + \partial_1 f(K_1, K_2) + \partial_2 f(K_1, K_2)g'(K_1)/(c\lambda + \beta).
\end{align*}
\]
Thus, it is easy to obtain the following result.

**Lemma 1.** (i) there exists a positive constant \(c_{\text{min}}\) such that
(a) if \(0 < c < c_{\text{min}}\) and \(\lambda \geq 0\), \(\Delta_1(c, \lambda) > 0\);
(b) if \(c \geq c_{\text{min}}\), the equation \(\Delta_1(c, \lambda) = 0\) has two positive real roots \(\lambda_1(c)\) and \(\lambda_2(c)\) with \(\lambda_1(c) \leq \lambda_2(c)\);
(c) if \(c = c_{\text{min}}\), then \(\lambda_1(c_{\text{min}}) = \lambda_2(c_{\text{min}}) := \lambda_\ast\); and if \(c > c_{\text{min}}\), then \(\lambda_1(c) < \lambda_\ast < \lambda_2(c)\), \(\lambda_1'(c) < 0\), \(\lambda_2'(c) > 0\), and
\[
\Delta_1(c, \lambda) = \begin{cases} > 0 & \text{for } \lambda \in \mathbb{R} \setminus (\lambda_1(c), \lambda_2(c)), \\
< 0 & \text{for } \lambda \in (\lambda_1(c), \lambda_2(c)). \end{cases}
\]
In particular, \(\frac{d}{dc}(c\lambda_1(c)) < 0\) for \(c > 0\);
(ii) the equation \(\Delta_2(c, \lambda) = 0\) has two real roots \(-\frac{\beta}{c} < \lambda_3(c) < 0\) and \(\lambda_4(c) > 0\).

2.1. Asymptotic behavior of traveling wave solutions. The following lemma is important to obtain the the asymptotic behavior of the wave profiles, which can be found in Carr and Chmaj [2].

**Lemma 2.** Let \(u(\xi)\) be a positive decreasing function and \(J_1(\Lambda) := \int_0^{+\infty} e^{-\Lambda \xi} u(\xi) d\xi\). If \(J_1(\Lambda)\) can be written as \(J_1(\Lambda) = J(\Lambda) + \Lambda_0^{-k+1}(\Lambda - \Lambda_0)^{-k+1}\), where \(k > -1\), \(\Lambda_0 > 0\) are two constants and \(J\) is analytic in the strip \(-\Lambda_0 \leq \text{Re}\Lambda < 0\), then
\[
\lim_{\xi \to +\infty} \frac{u(\xi)}{\xi^k e^{-\Lambda_0 \xi}} = \frac{J(-\Lambda_0)}{\Gamma(\Lambda_0 + 1)}.
\]

**Lemma 3.** Let \((\phi, \psi)\) be a traveling wave solution of \((3)\) with speed \(c \geq c_{\text{min}}\). Then,
(i) \(\phi(\xi) \in (0, K_1)\) and \(\psi(\xi) \in (0, K_2)\) for all \(\xi \in \mathbb{R}\);
(ii) \(\phi'(\pm\infty) = \psi'(\pm\infty) = 0\).
Proof. (i) Set \( L_1 = \max_{(u,v) \in [0,K]} |\partial_1 f(u,v)| \) and \( H(\phi)(\xi) = L_1 \phi(\xi) + f(\phi(\xi),\psi(\xi)) \).
From the first equation of (4), we have
\[
d\phi''(\xi) - c\phi'(\xi) - L_1 \phi(\xi) + H(\phi)(\xi) = 0. \quad (13)
\]
In view of \((\phi(\cdot),\psi(\cdot)) \in [0,K]\) and \(\partial_2 f(u,v) \geq 0\) for \((u,v) \in [0,K]\), we have \(H(\phi)(\xi) \geq f(\phi(\xi),0) + L_1 \phi(\xi) \geq 0\) for all \(\xi \in \mathbb{R}\).
By the linear ordinary differential equations theory, we obtain
\[
\phi(\xi) = \frac{1}{d(\lambda_4 - \lambda_3)} \left[ \int_{-\infty}^{\xi} e^{\lambda_3(\xi-s)} H(\phi)(s)ds + \int_{\xi}^{+\infty} e^{\lambda_4(\xi-s)} H(\phi)(s)ds \right], \quad (14)
\]
where
\[
\lambda_3 := \left( c - \sqrt{c^2 + 4L_1d} \right) / (2d) < 0 \quad \text{and} \quad \lambda_4 := \left( c + \sqrt{c^2 + 4L_1d} \right) / (2d) > 0.
\]
We first show that \(\phi(\cdot) > 0\) by a contradiction argument. Assume that there exists \(\xi_1 \in \mathbb{R}\) such that \(\phi(\xi_1) = 0\). Then
\[
0 = \phi(\xi_1) = \frac{1}{d(\lambda_4 - \lambda_3)} \left[ \int_{-\infty}^{\xi_1} e^{\lambda_3(\xi_1-s)} H(\phi)(s)ds + \int_{\xi_1}^{+\infty} e^{\lambda_4(\xi_1-s)} H(\phi)(s)ds \right].
\]
Since \(H(\phi)(\xi) \geq 0\) for all \(\xi \in \mathbb{R}\), \(H(\phi)(\xi) = 0\) for all \(\xi \in \mathbb{R}\), and hence \(0 = f(\phi(\xi),\psi(\xi)) + L_1 \phi(\xi) \geq \partial_2 f(\theta \phi(\xi),\theta \psi(\xi)) \psi(\xi) \geq 0\). According to \(\partial_2 f(u,v) > 0\) for \((u,v) \in [0,K]\) and \(\psi(\xi)\) satisfies (10), we have
\[
\psi(\xi) = \frac{1}{e} \int_{-\infty}^{\xi} e^{-\frac{c}{2}(\xi-s)} g(\phi(s))ds = 0 \quad \text{for} \quad \xi \in \mathbb{R}.
\]
Thus \(g(\phi(\xi)) = 0\) for all \(\xi \in \mathbb{R}\). Since \(g(0) = 0\) and \(g'(u) > 0\) for \(u \in [0,K_1]\), it must be \(\phi(\xi) = 0\) for all \(\xi \in \mathbb{R}\), which is a contradiction. Therefore \(\phi(\cdot) > 0\). Similarly, we can prove that \(\phi(\cdot) < K_1\).
In view of \(g(u) > 0\) for \(u \in (0,K_1]\) and \(\psi(\xi) = \frac{1}{e} \int_{-\infty}^{\xi} e^{-\frac{c}{2}(\xi-s)} g(\phi(s))ds\), it is easy to get \(\psi(\xi) \in (0,K_2)\) for all \(\xi \in \mathbb{R}\).
(ii) Clearly, \(\psi'(\pm\infty) = 0\). It follows from (14) and the L. Hospital’s rule that \(\phi'(\pm\infty) = 0\). This completes the proof. \(\square\)

Theorem 5. Assume that \((\phi(\xi),\psi(\xi))\) is a traveling wave solution of (3) with speed \(c \geq c_{\min}\). Then,
(i) for \(c > c_{\min}\),
\[
\lim_{\xi \to -\infty} \phi(\xi)e^{-\lambda_1(c)\xi} = a_0(c), \quad \lim_{\xi \to -\infty} \phi'(\xi)e^{-\lambda_1(c)\xi} = a_0(c)\lambda_1(c), \quad (15)
\]
\[
\lim_{\xi \to -\infty} \psi(\xi)e^{-\lambda_1(c)\xi} = A(c)a_0(c), \quad \lim_{\xi \to -\infty} \psi'(\xi)e^{-\lambda_1(c)\xi} = A(c)a_0(c)\lambda_1(c), \quad (16)
\]
and for \(c = c_{\min}\),
\[
\lim_{\xi \to -\infty} \phi(\xi)e^{-\lambda_1(c)\xi} = -a_0(c), \quad \lim_{\xi \to -\infty} \phi'(\xi)e^{-\lambda_1(c)\xi} = -a_0(c)\lambda_1(c), \quad (17)
\]
\[
\lim_{\xi \to -\infty} \psi(\xi)e^{-\lambda_1(c)\xi} = -A(c)a_0(c), \quad \lim_{\xi \to -\infty} \psi'(\xi)e^{-\lambda_1(c)\xi} = -A(c)a_0(c)\lambda_1(c), \quad (18)
\]
(ii) for \(c \geq c_{\min}\),
\[
\lim_{\xi \to +\infty} \left[ K_1 - \phi(\xi) \right] e^{-\lambda_3(c)\xi} = a_1(c), \quad \lim_{\xi \to +\infty} \phi'(\xi)e^{-\lambda_3(c)\xi} = -a_1(c)\lambda_3(c), \quad (19)
\]
\[
\lim_{\xi \to +\infty} \left[ K_2 - \phi(\xi) \right] e^{-\lambda_3(c)\xi} = B(c)a_1(c), \quad \lim_{\xi \to +\infty} \psi'(\xi)e^{-\lambda_3(c)\xi} = -B(c)a_1(c)\lambda_3(c), \quad (20)
\]
where \( a_0(c), a_1(c) \) are positive constants, \( A(c) = \frac{g'(0)}{c \lambda_1(c) + \beta} > 0 \), and \( B(c) = \frac{g'(K_1)}{c \lambda_1(c) + \beta} > 0 \) for \( c \geq c_{\text{min}} \).

**Proof.** We only prove the assertion (i), since the assertion (ii) can be proved similarly. First, we show that (15) and (17) hold. Note that \( \phi(\cdot) \in [0, K_1] \), \( \phi(-\infty) = 0 \), \( \phi(+\infty) = K_1 \) and \( \phi \) satisfies (11), i.e.,

\[
c \phi'(\xi) = d \phi''(\xi) + f \left( \phi(\xi), \frac{1}{c} \int_{-\infty}^{\xi} e^{-\frac{\gamma}{2}(\xi-s)g(\phi(s))} ds \right).
\]

The proofs of (15) and (17) are similar to those of [23, Theorem 4.8] and [24, Theorem 3.5], we only sketch the outline. The proof is divided into three steps.

**Step 1.** We show that \( \phi(\xi) \) is integrable on \((-\infty, \xi')\) for some \( \xi' \in \mathbb{R} \).

**Step 2.** We prove that \( \phi(\xi) = O(e^{\gamma \xi}) \) as \( \xi \to -\infty \) for some \( \gamma > 0 \). To get the assertion, we first show that \( W(\xi) = O(e^{\gamma \xi}) \) as \( \xi \to -\infty \), where \( W(\xi) := \int_{-\infty}^{\xi} \phi(\xi) \) ds.

**Step 3.** For \( 0 < \text{Re} \lambda < \gamma \), define a two-sided Laplace transform of \( \phi \) by \( \mathcal{L}(\lambda) = \int_{-\infty}^{+\infty} \phi(\xi)e^{-\lambda \xi} d\xi \). Using Lemma 2, one can show that for \( c > c_{\text{min}}, \lim_{\xi \to -\infty} \phi(\xi)e^{-\lambda_1(c)\xi} = a_0(c) \) and for \( c = c_{\text{min}}, \lim_{\xi \to -\infty} \phi(\xi)e^{-\lambda_1(c)\xi} = -a_0(c) \).

Integrating the two sides of the equality (21) from \(-\infty\) to \( \xi \), we obtain

\[
d \phi'(\xi) = c \phi(\xi) - \int_{-\infty}^{\xi} f \left( \phi(s), \frac{1}{c} \int_{-\infty}^{s} e^{-\frac{\gamma}{2}(s-r)g(\phi(r))} dr \right) ds.
\]

Then, it is easy to verify that for \( c > c_{\text{min}}, \lim_{\xi \to -\infty} \phi'(\xi)e^{-\lambda_1(c)\xi} = a_0(c)\lambda_1(c) \) and for \( c = c_{\text{min}}, \lim_{\xi \to -\infty} \phi'(\xi)e^{-\lambda_1(c)\xi} = -a_0(c)\lambda_1(c) \). Therefore, (15) and (17) hold.

Next, we prove (16) and (18). Since \( g \in C^2([0, K_1], \mathbb{R}) \) and \( g(u) \leq g'(0)u \) for \( u \in [0, K_1] \), \( \lim_{\xi \to -\infty} g(\phi(\xi))e^{-\lambda_1(c)\xi} = g'(0)a_0(c) \) for any \( c > c_{\text{min}} \). Hence, for \( c > c_{\text{min}} \),

\[
\lim_{\xi \to -\infty} \psi(\xi)e^{-\lambda_1(c)\xi} = \lim_{\xi \to -\infty} \frac{\int_{-\infty}^{\xi} e^{\frac{\gamma}{2}g(\phi(s))} ds}{ce^{(\lambda_1(c)+\frac{\gamma}{2})\xi}} = \frac{g'(0)}{c \lambda_1(c) + \beta} a_0(c) = A(c)a_0(c).
\]

It follows from the second equation of (4) that \( \lim_{\xi \to -\infty} \psi'(\xi)e^{-\lambda_1(c)\xi} = A(c)a_0(c)\lambda_1(c) \) for \( c > c_{\text{min}} \). Therefore, (16) holds. Similarly, one can show that (18) holds. This completes the proof.

**Corollary 1.** Let the assumptions of Theorem 5 be satisfied. Then, for all \( c \geq c_{\text{min}} \),

\[
\lim_{\xi \to -\infty} \frac{\phi'(\xi)}{\phi(\xi)} = \lim_{\xi \to -\infty} \frac{\psi'(\xi)}{\psi(\xi)} = \lambda_1(c)
\]

and

\[
\lim_{\xi \to +\infty} \frac{\phi'(\xi)}{\phi(\xi)} - K_1 = \lim_{\xi \to +\infty} \frac{\psi'(\xi)}{\psi(\xi)} - K_2 = \lambda_3(c).
\]

2.2. Monotonicity and uniqueness of traveling wave solutions. We first transform the monotonicity and uniqueness of traveling wave solutions of (3) to those of solutions of the scalar equation (11).

**Lemma 4.** Let \((\phi, \psi)\) be a traveling wave solution of (3) with speed \( c \geq c_{\text{min}} \). Then the following statements hold:
(i) if \( \phi'(\xi) \geq 0 \) for all \( \xi \in \mathbb{R} \), then \( \phi'(\xi) > 0 \) for all \( \xi \in \mathbb{R} \);
(ii) if \( \phi'(\xi) > 0 \) for all \( \xi \in \mathbb{R} \), then \( \psi'(\xi) > 0 \) for all \( \xi \in \mathbb{R} \).

**Proof.** (i) Note that

\[
\psi'(\xi) = \frac{1}{c} \int_{-\infty}^{\xi} \beta e^{-\frac{\beta}{c}(\xi-s)} [g(\phi(s)) - g(\phi(s))] ds.
\]  

(22)

Since \( \phi'(\xi) \geq 0 \) for all \( \xi \in \mathbb{R} \) and \( g'(u) > 0 \) for \( u \in [0, K_1] \), \( \psi'(\xi) \geq 0 \) for all \( \xi \in \mathbb{R} \).

Suppose for the contrary that there exists \( \xi_2 \in \mathbb{R} \) such that \( \phi'(\xi_2) = 0 \). Then, from (14), we get

\[
0 = d(\lambda_4 - \lambda_3) \phi'(\xi_2) = \left\{ \int_{-\infty}^{\xi_2} e^{\lambda_4(\xi_2-s)} \lambda_3 + \int_{\xi_2}^{+\infty} e^{\lambda_3(\xi_2-s)} \lambda_4 \right\} [H(\phi)(s) - H(\phi)(\xi_2)] ds.
\]

Since \( \lambda_3 < 0 \), \( \lambda_4 > 0 \), and \( H(\phi)(\xi) \) is increasing for \( \xi \in \mathbb{R} \), it must be \( H(\phi)(\xi) = H(\phi)(\xi_2) \) for all \( \xi \in \mathbb{R} \). Hence, for \( \xi \geq \xi_2 \),

\[
0 \geq \partial_2 f(\phi(\xi) - \psi(\xi_2)) + L_1[\phi(\xi) - \psi(\xi_2)]
\]

where \( \theta_1 \in (0, 1) \). In view of \( \partial_2 f(u, v) > 0 \) for \( (u, v) \in [0, K_1] \), we have \( \psi(\xi) = \psi(\xi_2) \) for all \( \xi \geq \xi_2 \). Letting \( \xi \to +\infty \), we get \( \psi(\xi_2) = K_2 \). However, \( \psi(\xi) < K_2 \) for all \( \xi \in \mathbb{R} \) according to Lemma 3. This contradiction implies that \( \phi'(\xi) > 0 \) for \( \xi \in \mathbb{R} \).

(ii) Note that \( g(0) = 0 \) and \( g'(u) > 0 \) for \( u \in [0, K_1] \). If there exists \( \xi_3 \in \mathbb{R} \) such that \( \phi'(\xi_3) = 0 \), it follows from (22) that \( g(\phi(s)) = g(\phi(\xi_3)) \) for all \( s \leq \xi_3 \) which implies that \( g(\phi(\xi_3)) = \lim_{s \to -\infty} g(\phi(s)) = 0 \). Thus, \( \phi(\xi_3) = 0 \), which is a contradiction. Therefore, \( \psi'(\xi) > 0 \) for all \( \xi \in \mathbb{R} \). This completes the proof. □

**Lemma 5.** Assume that \( (\phi_1, \psi_1) \) and \( (\phi_2, \psi_2) \) are traveling wave solutions of (3) with speed \( c \geq c_{\text{min}} \). If there exists \( \xi_0 \in \mathbb{R} \) such that \( \phi_1(\cdot + \xi_0) \equiv \phi_2(\cdot) \), then \( \psi_1(\cdot + \xi_0) \equiv \psi_2(\cdot) \).

The proof is easy, so we omit it.

From Lemmas 4 and 5, to prove the monotonicity and uniqueness of traveling wave solutions of system (3), it suffices to prove those of solutions of equation (11).

**Lemma 6.** Assume that \( \phi_1 \) and \( \phi_2 \) are two solutions of (11) and (12) with \( c \geq c_{\text{min}} \). If \( \phi_1 \geq \phi_2 \) on \( \mathbb{R} \), then either \( \phi_1 \equiv \phi_2 \) or \( \phi_1 > \phi_2 \) on \( \mathbb{R} \).

**Proof.** Suppose that there exists \( \xi_3 \in \mathbb{R} \) such that \( \phi_1(\xi_3) = \phi_2(\xi_3) \). Then, from (14),

\[
0 = d(\lambda_4 - \lambda_3) [\phi_1(\xi_3) - \phi_2(\xi_3)]
\]

\[
= \left\{ \int_{-\infty}^{\xi_3} e^{\lambda_4(\xi_3-s)} + \int_{\xi_3}^{+\infty} e^{\lambda_3(\xi_3-s)} \right\} [H(\phi_1)(s) - H(\phi_2)(s)] ds.
\]

As \( H(\phi_1)(\cdot) \geq H(\phi_2)(\cdot) \), \( H(\phi_1)(\xi) = H(\phi_2)(\xi) \) for all \( \xi \in \mathbb{R} \). Similar to the proof of Lemma 3, one can obtain \( g(\phi_1(\xi)) = g(\phi_2(\xi)) \) for all \( \xi \in \mathbb{R} \). Since \( g'(u) > 0 \) for all \( u \in [0, K_1] \), we get \( \phi_1 \equiv \phi_2 \). The proof is complete. □

**Lemma 7.** Let \( \phi \) be a solution of (11) and (12) with \( c \geq c_{\text{min}} \). Then \( \phi'(\cdot) \geq 0 \) on \( \mathbb{R} \).

**Proof.** Due to Corollary 1, we know that \( \phi \) is strictly increasing in \( \mathbb{R} \setminus [-N, N] \) for some \( N \gg 1 \). By Lemma 6 and using the sliding method similar to that of [3, Lemma 4.3], one can easily show that \( \phi'(\xi) \geq 0 \) for \( \xi \in \mathbb{R} \). This completes the proof. □
The monotonicity of traveling wave solutions of (3), i.e, Theorem 1(i), is a direct consequence of Lemmas 4 and 7.

The following two lemmas are important to prove the uniqueness of solutions of (11).

Lemma 8. Assume that \( \phi \) is a solution of (11) and (12) with \( c \geq c_{\min} \). Then there exists a constant \( \rho_0 := \rho_0(c, f) \in (0, 1) \) such that for any \( \rho \in (0, \rho_0) \),

\[
\begin{align*}
B(\rho, \xi) & := (1 + \rho) f \left( \phi(\xi), \frac{1}{c} \int_{-\infty}^{0} e^{\frac{\xi}{s}} g(\phi(s + \xi)) ds \right) \\
& - f \left( (1 + \rho) \phi(\xi), \frac{1}{c} \int_{-\infty}^{0} e^{\frac{\xi}{s}} g((1 + \rho) \phi(s + \xi)) ds \right) > 0
\end{align*}
\]
on \( \{ \xi | \phi(\xi) > K_1 - \rho_0 \} \).

Proof. Let \( \psi(\xi) = \frac{1}{c} \int_{-\infty}^{0} e^{\frac{\xi}{s}} g(\phi(s + \xi)) ds \). Since \( B(0, \xi) = 0 \) for all \( \xi \in \mathbb{R} \) and

\[
B_{\rho}(\rho, \xi) = f(\phi(\xi), \psi(\xi)) - \partial_1 f(\phi(\xi), \psi(\xi)) \phi(\xi)
\]

\[
- \partial_2 f(\phi(\xi), \psi(\xi)) \frac{1}{c} \int_{-\infty}^{0} e^{\frac{\xi}{s}} g(\phi(s + \xi)) \phi(s + \xi) ds
\]

\[
\to \to [\partial_1 f(K_1, K_2) + g'(K_1) \partial_2 f(K_1, K_2) / \beta] K_1 > 0,
\]
as \( \xi \to +\infty \), it is easy to see that the assertion holds. \( \square \)

For a given solution \( \phi \) of (11) and (12), we define

\[
\kappa = \kappa(\phi) := \sup \left\{ \phi(\xi) \middle| \phi(\xi) \leq K_1 - \rho_0 \right\}.
\]

Clearly, \( 0 < \kappa < +\infty \), since \( \lim_{\xi \to -\infty} \phi'(\xi) / \phi(\xi) = \lambda_1(c) > 0 \) and \( \phi'(\xi) > 0 \) on \( \mathbb{R} \).

Lemma 9. Assume that \( \phi_1 \) and \( \phi_2 \) are two solutions of (11) and (12) with \( c \geq c_{\min} \). If there exists \( \rho \in (0, \rho_0) \) such that \( (1 + \rho) \phi_1(\cdot - \kappa \rho) \geq \phi_2(\cdot) \) on \( \mathbb{R} \), where \( \kappa = \kappa(\phi_1) \), then \( \phi_1(\cdot) \geq \phi_2(\cdot) \) in \( \mathbb{R} \).

Proof. Define \( W(\rho, \xi) = (1 + \rho) \phi_1(\xi - \kappa \rho) - \phi_2(\xi) \) and \( \rho^* = \inf \{ \rho > 0 \mid W(\rho, \xi) \geq 0, \forall \xi \in \mathbb{R} \} \). By continuity, \( W(\rho^*, \xi) \geq 0 \) for all \( \xi \in \mathbb{R} \). It suffices to show that \( \rho^* = 0 \).

Suppose for the contrary that \( \rho^* \in (0, \rho_0) \). By the definition of \( \kappa \) and Theorem 1(i), we have

\[
\frac{d}{d\rho} W(\rho, \xi) = \phi_1(\xi - \kappa \rho) - \kappa(1 + \rho) \phi'_1(\xi - \kappa \rho) < 0,
\]
on \( \{ \xi | \phi_1(\xi - \kappa \rho) \leq K_1 - \rho_0 \} \). Also note that \( W(\rho^*, +\infty) = \rho^* K_1 > 0 \). Hence, there exists \( \xi_3 \) with \( \phi_1(\xi_3 - \kappa \rho^*) > K_1 - \rho_0 \) such that the function \( W(\rho^* \xi, \xi) \) attains its minimal at the point \( \xi_3 \), i.e., \( W(\rho^*, \xi_3) = W(\xi, \xi_3) = 0, W_{\xi_3}(\rho^*, \xi_3) \geq 0, \) and \( W(\rho^*, \xi) \geq 0 \) for \( \xi \in \mathbb{R} \). Then \( (1 + \rho^*) \phi_1(P_0) = \phi_2(\xi_3), (1 + \rho^*) \phi'_1(P_0) = \phi'_2(\xi_3), (1 + \rho^*) \phi''_1(P_0) \geq \phi''_2(\xi_3) \) and \( (1 + \rho^*) \phi_1(\xi - \kappa \rho^*) \geq \phi_2(\xi) \) for all \( \xi \in \mathbb{R} \), where
$P_0 = \xi_3 - \kappa \rho^*$. Hence, it follows from Lemma 8 that

$$
0 = c \phi_2'(\xi_3) - d \phi_2''(\xi_3) - f \left( \phi_2(\xi_3), \frac{1}{c} \int_{-\infty}^{0} e^{\frac{s}{\rho}} g \left( \phi_2(s + \xi_3) \right) ds \right)
$$

$$
\geq c(1 + \rho^*) \phi_1'(P_0) - d(1 + \rho^*) \phi_1''(P_0)
$$

$$
\left[ (1 + \rho^*) \phi_1'(P_0), \frac{1}{c} \int_{-\infty}^{0} e^{\frac{s}{\rho}} g \left( (1 + \rho^*) \phi_1(s + P_0) \right) ds \right]
$$

$$
> (1 + \rho^*) \left[ c \phi_1'(P_0) - d \phi_1''(P_0) - f \left( \phi(P_0), \frac{1}{c} \int_{-\infty}^{0} e^{\frac{s}{\rho}} g \left( \phi(s + P_0) \right) ds \right) \right] = 0.
$$

This contradiction implies that $\rho^* = 0$, and the assertion of the lemma follows. \hfill \Box

**Lemma 10.** Assume that $\phi_1$ and $\phi_2$ are two solutions of (11) and (12) with $c \geq c_{\min}$. Then there exists $\xi_0 \in \mathbb{R}$ such that $\phi_1(\cdot + \xi_0) \equiv \phi_2(\cdot)$.

**Proof.** By virtue of Lemmas 6 and 9 and applying the sliding method, the proof is similar to that of [3, Theorem 5.1]. We omit it here. \hfill \Box

The uniqueness of traveling wave solutions of (3), i.e., Theorem 1(ii), is a direct consequence of Lemmas 5 and 10.

3. **Existence of spatially independent solution.** In this section, we consider the existence and asymptotic behavior of the spatially independent solution $\Gamma(t) = (\Gamma_1(t), \Gamma_2(t))$ of (3) with

$$
\Gamma(-\infty) = 0 \quad \text{and} \quad \Gamma(+\infty) = K,
$$

that is, $\Gamma(t)$ satisfies

$$
\begin{cases}
\Gamma_1'(t) = f(\Gamma_1(t), \Gamma_2(t)), \\
\Gamma_2'(t) = -\beta \Gamma_2(t) + g(\Gamma_1(t)).
\end{cases}
$$

(24)

Throughout this section, we assume $(A_1)$-$(A_3)$. Using a similar argument as in section 2, we can transform the system (24) into a scalar differential equation with an integral term. In fact, from the second equation of (24) and $\Gamma_2(-\infty) = 0$, we obtain

$$
\Gamma_2(t) = \int_{-\infty}^{t} e^{-\beta(t-s)} g(\Gamma_1(s)) ds.
$$

(25)

Then, $\Gamma_1$ satisfies

$$
\Gamma_1'(t) = f \left( \Gamma_1(t), \int_{-\infty}^{t} e^{-\beta(t-s)} g(\Gamma_1(s)) ds \right).
$$

(26)

Conversely, if $\Gamma_1(t)$ is a non-decreasing solution of (26) with $\Gamma_1(-\infty) = 0$, and $\Gamma_1(+\infty) = K_1$, and $\Gamma_2(t)$ is defined by (25), then $(\Gamma_1(t), \Gamma_2(t))$ is a non-decreasing solution of (24), and satisfies (23).

For any $\lambda \in \mathbb{C}$, define the function

$$
\Delta_3(\lambda) := \lambda^2 + (\beta - \partial_1 f(0, 0)) \lambda - (\beta \partial_1 f(0, 0) + g'(0) \partial_2 f(0, 0)).
$$

**Lemma 11.** The equation $\Delta_3(\lambda) = 0$ has two real roots $\lambda_1^* < 0$ and $\lambda^* > 0$. In particular, for any $c \geq c_{\min}$, $\lambda^* < c \lambda_1(c)$, where $c_{\min}$ and $\lambda_1(c)$ are defined as in Lemma 1.
Proof. We only prove $\lambda^* < c\lambda_1(c)$ for any $c \geq c_{\min}$. Note that
\[
c\lambda_1(c) - \lambda^* = d\lambda_1^2(c) + \partial_1 f(0,0) + \frac{\partial_2 f(0,0)g'(0)}{c\lambda_1(c) + \beta} - \lambda^* > \partial_1 f(0,0) + \frac{\partial_2 f(0,0)g'(0)}{c\lambda_1(c) + \beta} - \lambda^*.
\]
If there exists $c_0 \geq c_{\min}$ such that $\lambda^* \geq c_0\lambda_1(c_0)$, then
\[
0 \geq c_0\lambda_1(c_0) - \lambda^* > \partial_1 f(0,0) + \frac{\partial_2 f(0,0)g'(0)}{\lambda^* + \beta} - \lambda^* = 0,
\]
which is a contradiction. Hence $\lambda^* < c\lambda_1(c)$ for any $c \geq c_{\min}$.

We now consider the spaces $C(\mathbb{R}, \mathbb{R})$ of continuous real functions on $\mathbb{R}$, and the operator $T : C(\mathbb{R}, [0, K_1]) \rightarrow C(\mathbb{R}, \mathbb{R})$ defined by
\[
T(\phi)(t) = \int_{-\infty}^{t} e^{-L_1(t-s)} h(\phi)(s) ds,
\]
where $h(\phi)(t) = f(\phi(t), \int_{-\infty}^{t} e^{-\beta(t-s)}g(\phi(s)) ds) + L_1 \phi(t)$.

It is easy to see that the following result holds.

**Lemma 12.** (i) $T : C(\mathbb{R}, [0, K_1]) \rightarrow C(\mathbb{R}, [0, K_1])$;
(ii) $T(\phi)(t) \geq T(\psi)(t)$ for $\phi, \psi \in C(\mathbb{R}, [0, K_1])$ with $\phi(t) \geq \psi(t)$;
(iii) $T(\phi)(t)$ is increasing in $\mathbb{R}$ for $\phi \in C(\mathbb{R}, [0, K_1])$ with $\phi(t)$ is increasing in $\mathbb{R}$.

For any fixed $\epsilon \in (0, \min\{1, K_1\})$ and sufficiently large $q > 1$, define two functions as follows:
\[
\overline{\phi}(t) = \min\{K_1, e^{\lambda^* t}\} \quad \text{and} \quad \underline{\phi}(t) = \max\{0, (1-qe^{\lambda^* t}) e^{\lambda^* t}\}, \quad t \in \mathbb{R}.
\]
Direct computations show that the following result holds.

**Lemma 13.** (i) $0 \leq \phi(t) \leq \overline{\phi}(t) \leq K_1$ for all $t \in \mathbb{R}$;
(ii) $T(\overline{\phi})(t) \leq \overline{\phi}(t)$ and $T(\underline{\phi})(t) \geq \underline{\phi}(t)$ for all $t \in \mathbb{R}$.

**Proof of Theorem 2.** Using the monotone iteration technique, we can show that equation (26) admits a solution $\Gamma(t)$ which satisfies $\Gamma_1'(t) \geq 0$ for $t \in \mathbb{R}$ and $\underline{\phi}(t) \leq \Gamma_1(t) \leq \overline{\phi}(t)$ for all $t \in \mathbb{R}$. Thus, $\lim_{t \rightarrow -\infty} \Gamma_1(t)e^{-\lambda^* t} = 1$, $\Gamma_1(+\infty) \in (0, K_1]$ and $0 \leq \Gamma_1(t) \leq e^{\lambda^* t}$ for all $t \in \mathbb{R}$. Furthermore, one can easily get that $\Gamma_1(+\infty) = K_1$. Let
\[
\Gamma_2(t) = \int_{-\infty}^{t} e^{-\beta(t-s)}g(\Gamma_1(s)) ds.
\]
Obviously, $\Gamma_2(t)$ is non-decreasing, and satisfies $\Gamma_2(+\infty) = K_2$, $\lim_{t \rightarrow -\infty} \Gamma_2(t)e^{-\lambda^* t} = b_*$, and $0 \leq \Gamma_2(t) \leq b_* e^{\lambda^* t}$ for all $t \in \mathbb{R}$.

Therefore, $\Gamma(t) = (\Gamma_1(t), \Gamma_2(t))$ meets Theorem 2 except $\Gamma_1'(t) > 0$ and $\Gamma_2'(t) > 0$ for $t \in \mathbb{R}$. According to (24), we have
\[
\begin{cases}
\Gamma_1''(t) = \partial_1 f(\Gamma_1(t), \Gamma_2(t)) \Gamma_1'(t) + \partial_2 f(\Gamma_1(t), \Gamma_2(t)) \Gamma_2'(t), \\
\Gamma_2''(t) = -\beta \Gamma_2(t) + g'(\Gamma_1(t)) \Gamma_1'(t),
\end{cases}
\]
which implies that for any $\tau < t$,
\[
\Gamma'_1(t) = \Gamma'_1(\tau)e^{-L_1(t-\tau)} + \int_\tau^t R(s)e^{-L_1(t-s)}ds \geq \Gamma'_1(\tau)e^{-L_1(t-\tau)} \geq 0,
\]
\[
\Gamma'_2(t) = \Gamma'_2(\tau)e^{-\beta(t-\tau)} + \int_\tau^t g'(\Gamma_1(s))\Gamma'_1(s)e^{-\beta(t-s)}ds \geq \Gamma'_2(\tau)e^{-\beta(t-\tau)} \geq 0,
\]
where $R(t) = [\partial_1 f(\Gamma_1(t), \Gamma_2(t)) + L_1] \Gamma'_1(t) + \partial_2 f(\Gamma_1(t), \Gamma_2(t))\Gamma'_2(t) \geq 0$ for all $t \in \mathbb{R}$. Suppose for the contrary that there exists $t_0 \in \mathbb{R}$ such that $\Gamma'_1(t_0) = 0$, it then follows from (27) that $\Gamma'_1(\tau) = 0$ for all $\tau < t_0$, which contradicts the fact $\lim_{t \to -\infty} \Gamma_1(t)e^{-\lambda^+t} = 1$. Therefore, $\Gamma'_1(t) > 0$ for $t \in \mathbb{R}$. Similarly, we can prove that $\Gamma'_2(t) > 0$ for $t \in \mathbb{R}$. This completes the proof of Theorem 2.

**Remark 2.** (i) It should be point out the existence of $\Gamma(t)$ can also be established using the monotone dynamical systems theory. However, the exponential decay rate of the spatially independent solution at minus infinity can not be obtained using this method. (ii) To obtain the existence of the monotone spatially independent solution, the condition $g'(u) > 0$ for $u \in [0, K_1]$ and $\partial_2 f(u, v) > 0$ for $(u, v) \in [0, K]$ can be replaced by $g'(u) \geq 0$ for $u \in [0, K_1]$ and $\partial_2 f(u, v) \geq 0$ for $(u, v) \in [0, K]$.

### 4. Existence of entire solutions

In this section, we first state some definitions and comparison theorems, and give a priori estimate on solutions of (3). Then we prove the main results of entire solutions by using the comparison argument and constructing appropriate sub-super-solution pairs. Throughout this section, we always assume $(A_1)$, $(A_2)$, and $(A_3)^\prime$.

#### 4.1. Preliminaries

Let $X = \text{BUC}(\mathbb{R}, \mathbb{R}^2)$ be the Banach space of all bounded and uniformly continuous functions from $\mathbb{R}$ into $\mathbb{R}^2$ with the supremum norm $\| \cdot \|_X$. Let $X^+ = \{ \varphi = (\varphi_1, \varphi_2) \in X : \varphi_i(x) \geq 0, x \in \mathbb{R}, i = 1, 2 \}$. It is easy to see that $X^+$ is a closed cone of $X$. For any $\phi^1, \phi^2 \in X$, we write $\phi^1 \preceq_X \phi^2$ if $\phi^2 - \phi^1 \in X^+$. For $\phi^1, \phi^2 \in X$ with $\phi^1 \preceq_X \phi^2$, we denote $[\phi^1, \phi^2]_X = \{ \phi \in X : \phi^1 \preceq_X \phi \preceq_X \phi^2 \}$.

Let $T_2(t)\psi = e^{-\beta t}\psi$, $\forall \psi \in \text{BUC}(\mathbb{R}, \mathbb{R})$ and $T_1(t)$ be the analytic semigroup on $\text{BUC}(\mathbb{R}, \mathbb{R})$ generated by $u_t = du_{xx}$. Clearly, $T(t) = (T_1(t), T_2(t))$ is a linear semigroup on $X$.

**Definition 1.** A continuous function $w = (u, v) : [s, T) \to X, s < T$, is called a supersolution (or a subsolution) of (3) on $[s, T)$ if
\[
w(t) \geq (or \leq) T(t - \tau)w(\tau) + \int_\tau^t T(t - r)B(w(r))dr
\]
for any $s \leq \tau < t < T$, where $B(w) = (f(u, v), g(u))$.

A function $w : (-\infty, T) \to X$ is called a supersolution (or a subsolution) of (3) on $(-\infty, T)$, if for any $s < T$, $w$ is a supersolution (or a subsolution) of (3) on $[s, T)$.

Using the theory of abstract functional differential equations [18, Corollary 5], it is easy to prove that the following result holds, see e.g., [17, 28].

**Lemma 14.** (i) For any $\varphi \in [0, K]_X$, (3) has a unique classical solution $w(x, t; \varphi)$ on $(x, t) \in \mathbb{R} \times [0, \infty)$ with $w(x, 0; \varphi) = \varphi(x)$ and $0 \leq w(x, t; \varphi) \leq K$ for $x \in \mathbb{R}$, $t \geq 0$.

(ii) For any pair of supersolution $w^+(x, t)$ and subsolution $w^-(x, t)$ of (3) on $[0, \infty)$
with $0 \leq w^-(x,t), w^+(x,t) \leq K$ for $(x,t) \in \mathbb{R} \times [0, \infty)$, and $w^+(x,0) \geq w^-(x,0)$ for $x \in \mathbb{R}$, there holds $0 \leq w^-(x,t) \leq w^+(x,t) \leq K$ for $(x,t) \in \mathbb{R} \times [0, \infty)$.

The following result plays an important role in the proof of our main results.

**Lemma 15.** Suppose that $w(x,t) = (u(x,t), v(x,t))$ is a solution of (3) with initial value $\varphi = (\varphi_1, \varphi_2) \in [0,K]_x$, then there exists a positive constant $M > 0$, independent of $\varphi$, such that for any $\eta > 0$, $x \in \mathbb{R}$ and $t > 1$,

$$
|u_t(x,t)|, |u_{tx}(x,t)|, |u_x(x,t)| \leq M,
$$

$$
|u_{xt}(x,t)|, |u_{xx}(x,t)|, |u_{xxx}(x,t)|, |v_t(x,t)|, |v_{tt}(x,t)| \leq M.
$$

If, in addition, there exists $L' > 0$ such that for any $\eta > 0$, $x \in \mathbb{R}$ and $t > 1$,

$$
|u_{xx}(x+\eta,t) - u_{xx}(x,t)|, |v(x+\eta,t) - v(x,t)|, |v_t(x+\eta,t) - v_t(x,t)| \leq M'\eta,
$$

where $M' > 0$ is a constant which is independent of $\varphi$ and $\eta$.

**Proof.** From Lemma 14, we see that $0 \leq (u(x,t), v(x,t)) \leq K$ for all $x \in \mathbb{R}$ and $t \geq 0$. By the second equation of (3), we have for $x \in \mathbb{R}$ and $t \geq 0$,

$$
|v_t(x,t)| \leq \beta K_2 + g(K_1) := M_1.
$$

Note that for any $s \geq 0$ and $t > s$,

$$
u(x,t) = \int_{-\infty}^{+\infty} e^{-(x-y)^2/(4\pi t-s)} u(y,s)dy + \int_s^t \int_{-\infty}^{+\infty} e^{-(x-y)^2/(4\pi (t-r))} f(u(y,r),v(y,r))dydr.
$$

Consequently, for any $s \geq 0$ and $t \in [s+1, s+4],$

$$
u_x(x,t) = \int_{-\infty}^{+\infty} \frac{-(x-y)}{2d(t-s)} e^{-(x-y)^2/(4\pi (t-s))} u(y,s)dy
$$

$$
+ \int_s^t \int_{-\infty}^{+\infty} \frac{-(x-y)}{2d(t-r)} e^{-(x-y)^2/(4\pi (t-r))} f(u(y,r),v(y,r))dydr.
$$

Similar to the proof of [24, Proposition 4.3], we can easily verify that

$$
|u_x(x,t)| \leq \frac{K_1}{\sqrt{\pi d(t-s)}} + \frac{2\sqrt{(t-s)}}{\sqrt{\pi d}} L_2 \leq \frac{K_1}{\sqrt{\pi d}} + \frac{4L_2}{\sqrt{\pi d}} := M_2,
$$

where $L_2 = \max_{(x,u) \in [0,K]} |f(u,v)|$. Since $s \geq 0$ is arbitrary, which implies that $|u_x(x,t)| \leq M_2$ for any $x \in \mathbb{R}$ and $t > 1$.

Using the estimate for $v_t$ and applying a similar argument, we can find a positive constant $M_3$, which is independent of $\varphi$, such that for any $x \in \mathbb{R}$ and $t > 1$,

$$
|u_t(x,t)|, |u_{tx}(x,t)|, |u_{xt}(x,t)|, |u_{xx}(x,t)|, |u_{tt}(x,t)|, |u_{xtt}(x,t)| \leq M_3.
$$

Then, for any $x \in \mathbb{R}$ and $t > 1$,

$$
|v_{tt}(x,t)| = | - \beta v_t(x,t) + g'(u(x,t))u_t(x,t)| \leq \beta M_1 + M_3 \max_{u \in [0,K]} |g'(u)| := M_4.
$$

Take $M = \max\{M_1, \cdots, M_4\}$. Then the first statement of this lemma follows.

Now we prove (29). Note that

$$
v(x,t) = \varphi_2(x)e^{-\beta t} + \int_0^t g(u(x,s))e^{-\beta(t-s)}ds, \forall x \in \mathbb{R}, t > 0.
$$
By our assumption, we have for any \( \eta > 0, x \in \mathbb{R} \) and \( t > 1 \),

\[
|v(x + \eta, t) - v(x, t)| \leq |\varphi_2(x + \eta) - \varphi_2(x)| + \int_0^t \left| g(u(x + \eta, s) - g(u(x, s)) \right| e^{-\beta(t-s)} ds \\
\leq L' \eta + \frac{M}{\beta} \max_{u \in [0,K_1]} g'(u) \eta := M_1' \eta.
\]

Moreover, for any \( \eta > 0, x \in \mathbb{R} \) and \( t > 1 \),

\[
|u_t(x + \eta, t) - u_t(x, t)| \leq \beta |v(x + \eta, t) - v(x, t)| + |g(u(x + \eta, t) - g(u(x, t))| \\
\leq [\beta M_1' + M \max_{u \in [0,K_1]} g'(u)] \eta := M_2' \eta,
\]

and

\[
|u_{xx}(x + \eta, t) - u_{xx}(x, t)| \leq \frac{1}{d} |u_t(x + \eta, t) - u_t(x, t)| \\
+ \frac{1}{d} |f(u(x + \eta, t), v(x + \eta, t)) - f(u(x, t), v(x, t))| \\
\leq \frac{1}{d} [M + LM + LM'] \eta := M_3' \eta.
\]

Here \( L = \max_{(u,v) \in [0,1]} \{ |\partial_i f(u,v)| \} i = 1, 2 \). Letting \( M' = \max \{ M_1', M_2', M_3' \} \), (29) holds. This completes the proof. \( \square \)

**Lemma 16.** Assume that \( w^+ = (u^+, v^+) \in C(\mathbb{R} \times [0, +\infty), [0, +\infty)^2) \) and \( w^- = (u^-, v^-) \in C(\mathbb{R} \times [0, +\infty), (-\infty, K_1] \times (-\infty, K_2]) \) satisfy \( w^+(x, 0) \geq w^-(x, 0) \) for \( x \in \mathbb{R} \), and

\[
\begin{cases}
    u^+_t(x, t) \geq du^+_x(x, t) + \partial_1 f(0, 0) u^+(x, t) + \partial_2 f(0, 0) v^+(x, t), \\
    v^+_t(x, t) \geq -\beta v^+(x, t) + g'(0) u^+(x, t), \\
    u^-_t(x, t) \leq du^-_x(x, t) + \partial_1 f(0, 0) u^-(x, t) + \partial_2 f(0, 0) v^-(x, t), \\
    v^-_t(x, t) \leq -\beta v^-(x, t) + g'(0) u^-(x, t),
\end{cases}
\]

for \( x \in \mathbb{R} \) and \( t > 0 \). Then \( w^+(x, t) \geq w^-(x, t) \) for all \( x \in \mathbb{R} \) and \( t \geq 0 \).

**Proof.** Set \( w(x, t) = (w_1(x, t), w_2(x, t)) := w^+(x, t) - w^-(x, t) \) for \( x \in \mathbb{R} \) and \( t \geq 0 \), then \( w(x, t) \) satisfies \( w(x, 0) \geq 0 \) and

\[
\begin{align*}
    w_{1,t}(x, t) &\geq dw_{1,x}(x, t) + \partial_1 f(0, 0) w_1(x, t) + \partial_2 f(0, 0) w_2(x, t), \\
    w_{2,t}(x, t) &\geq -\beta w_2(x, t) + g'(0) w_1(x, t),
\end{align*}
\]

for \( x \in \mathbb{R} \) and \( t > 0 \). From (30), we get

\[
w_1(x, t) \geq \int_{-\infty}^{+\infty} J(x-y, t) w_1(y, 0) dy + \partial_2 f(0, 0) \int_0^t \int_{-\infty}^{+\infty} J(x-y, t-s) w_2(y, s) dy ds \\
\geq \partial_2 f(0, 0) \int_0^t \int_{-\infty}^{+\infty} J(x-y, t-s) w_2(y, s) dy ds,
\]

for all \( x \in \mathbb{R} \) and \( t \geq 0 \), where \( J(x, t) = \frac{1}{\sqrt{4\pi t}} \exp \{ -\frac{x^2}{4t} + \partial_1 f(0, 0) t \} \), which implies that

\[
w_{2,t}(x, t) \geq -\beta w_2(x, t) + g'(0) \partial_2 f(0, 0) \int_0^t \int_{-\infty}^{+\infty} J(x-y, t-s) w_2(y, s) dy ds.
\]

Using the same method as that in [25, Theorem 3.4], it is easy to prove that \( w_2(x, t) \geq 0 \) for all \( x \in \mathbb{R} \) and \( t \geq 0 \). It then follows from (32) that \( w_1(x, t) \geq 0 \).
for all $x \in \mathbb{R}$ and $t \geq 0$. Therefore, $w^+(x, t) \geq w^-(x, t)$ for $x \in \mathbb{R}$ and $t \geq 0$. This completes the proof. \hfill $\square$

4.2. Proof of Theorem 3. Let $\Phi_c(\cdot) = (\phi_c(\cdot), \psi_c(\cdot))$ be a traveling wave solution of (3) with speed $c \geq c_{\text{min}}$. By Theorems 1 and 5, $\phi_c(\xi) > 0$ and $\psi_c(\xi) > 0$ for all $\xi \in \mathbb{R}$ and there exist positive constants $k(c), K(c), \mu(c), \eta(c)$ such that for $c > c_{\text{min}}$,

$$k(c)e^{\lambda_1(c)x} \leq \phi_c(x) \leq K(c)e^{\lambda_1(c)x}, \ x \leq 0,$$  (34)

$$\mu(c)\phi_c(x) \leq \psi_c(x) \leq \eta(c)\phi_c(x), \ x \leq 0,$$  (35)

$$\mu(c)\phi_c(x) \leq \phi'_c(x), \ \mu(c)\psi_c(x) \leq \psi'_c(x), \ x \leq 0,$$  (36)

and for $c \geq c_{\text{min}}$,

$$k(c)e^{\lambda_3(c)x} \leq K_1 - \phi_c(x) \leq K(c)e^{\lambda_1(c)x}, \ x \geq 0,$$  (37)

$$\mu(c)[K_1 - \phi_c(x)] \leq K_2 - \psi_c(x) \leq \eta(c)[K_1 - \phi_c(x)], \ x \geq 0.$$  (38)

For $c = c_{\text{min}}$, let $c \in (0, \lambda_+)$, there exists $K_\epsilon > 0$ such that

$$\phi_{c_{\text{min}}}(x) \leq K_\epsilon e^{(\lambda_+ - \epsilon)x}, \ x \leq 0.$$  

Also, there exist constants $\mu(c_{\text{min}}) > 0$ and $\eta(c_{\text{min}}) > 0$ such that

$$\mu(c_{\text{min}})\phi_{c_{\text{min}}}(x) \leq \psi_{c_{\text{min}}}(x) \leq \eta(c_{\text{min}})\phi_{c_{\text{min}}}(x), \ x \leq 0,$$

$$\mu(c_{\text{min}})\phi_{c_{\text{min}}}(x) \leq \phi'_{c_{\text{min}}}(x), \ \mu(c_{\text{min}})\psi_{c_{\text{min}}}(x) \leq \psi'_{c_{\text{min}}}(x), \ x \leq 0.$$  

Motivated by [9, 16], we consider the coupled system of ordinary differential equations

$$\begin{cases}
p'_1(t) = c_1 + Ne^{\alpha p_1(t)}, & t < 0, \\
p'_2(t) = c_2 + Ne^{\alpha p_2(t)}, & t < 0, \\
p_1(0) \leq 0, \ p_2(0) \leq 0,
\end{cases}$$  (39)

where $c_1, c_2, N, \alpha$ are positive constants and $c_2 \geq c_1 \geq c_{\text{min}}$. Solving the equation explicitly, we get

$$p_i(t) = p_i(0) + c_t - \frac{1}{\alpha} \ln \left(1 + \frac{N}{c_1} e^{\alpha p_i(0)} \right), \ i = 1, 2.$$  

Obviously, $p_i(t)$ is increasing, $i = 1, 2$, and $p_2(t) \leq p_1(t)$ for $t \leq 0$, if $p_2(0) \leq p_1(0)$. Let

$$\omega_1 = p_1(0) - \frac{1}{\alpha} \ln \left(1 + \frac{N}{c_1} e^{\alpha p_1(0)} \right) \text{ and } \omega_2 = p_2(0) - \frac{1}{\alpha} \ln \left(1 + \frac{N}{c_1} e^{\alpha p_1(0)} \right).$$  (40)

According to the identity $p_i(t) - c_t - \omega_i = \frac{1}{\alpha} \ln \left(1 + \frac{N}{c_1} e^{\alpha p_i(0)} \right)$, we have

$$0 < p_1(t) - c_1 t - \omega_1 = p_2(t) - c_2 t - \omega_2 \leq R_0 e^{c_{1} t} \text{ for } t \leq 0,$$

where $r = \frac{N}{c_1} e^{\alpha p_1(0)}$ and $R_0$ is a positive constant.

We first construct a subsolution of (3). The proof is similar to that of [27, Lemma 3.5], see also [16, Lemma 3.6]. We omit it here.

**Lemma 17.** The function $w(x, t) = (u(x, t), v(x, t))$ defined by

$$w(x, t) = \max \{\Phi_{c_1}(x + c_1 t + \omega_1), \Phi_{c_2}(-x + c_2 t + \omega_2)\}$$

is a subsolution of (3) on $(-\infty, +\infty)$. 
In order to construct supersolutions of (3), we make the following extension for the function $g$. Let $\sigma > 1$ be a constant. Define $\hat{g} : [0, \sigma K_1] \to \mathbb{R}$ by

$$
\hat{g}(u) = \begin{cases} 
g(u), & u \in [0, K_1], 
g(K_1) + g'(K_1)(u - K_1), & u \in [K_1, \sigma K_1].
\end{cases}
$$

Obviously, $\hat{g} \in C^1([0, \sigma K_1])$, $0 < g'(u) \leq g'(0)$ for $u \in [0, \sigma K_1]$ and $|g'(u_1) - g'(u_2)| \leq L_2|u_1 - u_2|$ for all $u_1, u_2 \in [0, \sigma K_1]$, where $L_2 = \max_{u \in [0, K_1]} |g''(u)|$. For convenience, we denote $\hat{g}$ by $g$ in the remainder of this paper.

**Lemma 18.** Let $L = \max_{(u,v) \in [0,K]} \{ |\partial_{ij}f(u,v)| |i,j=1,2 \}$ and choose positive constants $N$ and $\alpha$ in (39) as follows:

(i): if $c_2 \geq c_1 > c_{\min}$, let $\alpha = \lambda_1(c_2)$ and

$$
N \geq \max \left\{ \frac{L_2 K_1}{\mu^2(c_1)}, \frac{L_2 K_2}{\mu^2(c_2)}, \frac{L K_1}{\mu(c_1)} \left[ 1 + \eta(c_1) \right]^2, \frac{L K_2}{\mu(c_2)} \left[ 1 + \eta(c_2) \right]^2 \right\};
$$

(ii): if $c_2 > c_1 = c_{\min}$, let $\alpha = \lambda_1(c_2)$ and

$$
N \geq \max \left\{ \frac{L_2 K_2}{\mu^2(c_{\min})}, \frac{L K_2}{\mu(c_{\min})} \left[ 1 + \eta(c_{\min}) \right]^2, \frac{L K_2}{\mu(c_2)} \left[ 1 + \eta(c_2) \right]^2 \right\},
$$

for some $\epsilon \in (0, \lambda_1 - \lambda_1(c_2))$ with $\lambda_* = \lambda_1(c_{\min})$;

(iii): if $c_2 = c_1 = c_{\min}$, let $\alpha = \lambda_1(c_{\min})$ and

$$
N \geq L \max \left\{ \frac{L_2 K_2}{\mu^2(c_{\min})}, \frac{L K_2}{\mu(c_{\min})} \left[ 1 + \eta(c_{\min}) \right]^2 \right\},
$$

for some $\epsilon \in (0, \lambda_1)$.

Then, for the solution $(p_1(t), p_2(t))$ of (39) with $p_2(0) \leq p_1(0) \leq 0$, there exists $T < 0$ such that the function $\varpi(x,t) = (\varpi(x,t), \overline{\varpi}(x,t))$ defined by

$$
\varpi(x,t) = \min \left\{ K, \Phi_{c_1}(x + p_1(t)) + \Phi_{c_2}(-x + p_2(t)) \right\}
$$

is a supersolution of (3) on $(-\infty, T]$.

**Proof.** We only consider the case $c_2 \geq c_1 > c_{\min}$, since the other cases can be discussed similarly. For convenience, we denote $\mathbb{L} \{ \varpi(x,t) \} = (\mathbb{L}_1[\varpi(x,t)], \mathbb{L}_2[\varpi(x,t)])$, where

$$
\mathbb{L}_1[\varpi(x,t)] := \frac{\partial \varpi}{\partial t} - \partial^2 \varpi = f(\varpi(x,t), \overline{\varpi}(x,t))
$$

and

$$
\mathbb{L}_2[\varpi(x,t)] := \frac{\partial \varpi}{\partial t} + \beta \varpi(x,t) - g(\varpi(x,t)).
$$

Define

$$
S_1 = \left\{ (x,t) \in \mathbb{R} \times (-\infty, 0) \left| \phi_{c_1}(x + p_1(t)) + \phi_{c_2}(-x + p_2(t)) > K_1 \right. \right\},
$$

$$
S_2 = \left\{ (x,t) \in \mathbb{R} \times (-\infty, 0) \left| \phi_{c_1}(x + p_1(t)) + \phi_{c_2}(-x + p_2(t)) < K_1 \right. \right\},
$$

$$
S_3 = \left\{ (x,t) \in \mathbb{R} \times (-\infty, 0) \left| \psi_{c_1}(x + p_1(t)) + \psi_{c_2}(-x + p_2(t)) > K_2 \right. \right\},
$$

$$
S_4 = \left\{ (x,t) \in \mathbb{R} \times (-\infty, 0) \left| \psi_{c_1}(x + p_1(t)) + \psi_{c_2}(-x + p_2(t)) < K_2 \right. \right\}.
$$

The rest of the proof is divided into three steps.

**Step 1.** We verify that $\mathbb{L}_1[\varpi](x,t) \geq 0$ for $(x,t) \in S_1 \cup S_2$.

**Case (1).** For $(x,t) \in S_1$, $\varpi(x,t) = K_1$ and $\overline{\varpi}(x,t) \leq K_2$. Then, according to $\partial_2f(u,v) \geq 0$ for $(u,v) \in [0,K]$, we obtain $\mathbb{L}_1[\varpi](x,t) = -f(K_1, \overline{\varpi}(x,t)) \geq 0$, which implies $\mathbb{L}_1[\varpi](x,t) \geq 0$. For $(x,t) \in S_2$, the proof is similar.
\(-f(K_1, K_2) = 0.\)

**Case (II).** For \((x, t) \in S_2, \overline{\nu}(x, t) = \phi_{c_1}(x + p_1(t)) + \phi_{c_2}(-x + p_2(t)).\) Then,

\[
\mathbb{L}_1[\overline{\nu}](x, t) = p_1' \phi'_{c_1} + p_2' \phi'_{c_2} - d \phi''_{c_1} - d \phi''_{c_2} - f(\phi_{c_1} + \phi_{c_2}, \overline{\nu})
= (p_1' - c_1) \phi'_{c_1} + (p_2' - c_2) \phi'_{c_2} + f(\phi_{c_1}, \psi_{c_1}) + f(\phi_{c_2}, \psi_{c_2}) - f(\phi_{c_1} + \phi_{c_2}, \overline{\nu})
= [\phi'_{c_1} + \phi'_{c_2}] [Ne^{\lambda_1(t)} p_1(t) - H_1(x, t)],
\]

where

\[
H_1(x, t) = \frac{G_1(x, t)}{\phi'_{c_1}(x + p_1(t)) + \phi'_{c_2}(-x + p_2(t))}
\]

and

\[
G_1(x, t) = f(\phi_{c_1} + \phi_{c_2}, \overline{\nu}) - f(\phi_{c_1}, \psi_{c_1}) - f(\phi_{c_2}, \psi_{c_2}). \tag{41}
\]

We claim that

\[
G_1(x, t) \leq L \left[ \phi_{c_1} \left( \left(-1\right)^{t-1}x + p_t(t) \right) + \psi_{c_1} \left( \left(-1\right)^{t-1}x + p_t(t) \right) \right]^2
\]

for \((x, t) \in \mathbb{R} \times (-\infty, 0],\) where \(i = 1, 2.\) In fact, if \(\overline{\nu}(x, t) = K_2,\) i.e., \(K_2 - \psi_{c_1}(x + p_1(t)) \leq \psi_{c_2}(-x + p_2(t)),\) then, by \(\partial_t f(u, v) \leq \partial_x f(0, 0)\) for \((u, v) \in [0, K],\) \(i = 1, 2,\) we have

\[
G_1(x, t) = \int_0^1 \left[ \partial_t f(\phi_{c_1} + \theta \phi_{c_2}, \theta K_2 + (1 - \theta) \psi_{c_1}) \phi_{c_2}
+ \partial_2 f(\phi_{c_1} + \theta \phi_{c_2}, \theta K_2 + (1 - \theta) \psi_{c_1}) (K_2 - \psi_{c_1}) \right] d\theta
- \int_0^1 \left[ \partial_1 f(\theta \phi_{c_2}, \theta \psi_{c_2}) \phi_{c_2} + \partial_2 f(\theta \phi_{c_2}, \theta \psi_{c_2}) \psi_{c_2} \right] d\theta
\leq \int_0^1 \left\{ \left[ \partial_1 f(0, 0) - \partial_1 f(\theta \phi_{c_2}, \theta \psi_{c_2}) \right] \phi_{c_2} + \left[ \partial_2 f(0, 0) - \partial_2 f(\theta \phi_{c_2}, \theta \psi_{c_2}) \right] \psi_{c_2} \right\} d\theta
\leq L \left[ \phi_{c_2}(-x + p_2(t)) + \psi_{c_2}(-x + p_2(t)) \right]^2.
\]

If \(\overline{\nu}(x, t) = \psi_{c_1}(x + p_1(t)) + \psi_{c_2}(-x + p_2(t)),\) then we have

\[
G_1(x, t) = \int_0^1 \left[ \partial_1 f(\phi_{c_1} + \theta \phi_{c_2}, \psi_{c_1} + \theta \psi_{c_2}) \phi_{c_2} + \partial_2 f(\phi_{c_1} + \theta \phi_{c_2}, \psi_{c_1} + \theta \psi_{c_2}) \psi_{c_2} \right] d\theta
- \int_0^1 \left[ \partial_1 f(\theta \phi_{c_2}, \theta \psi_{c_2}) \phi_{c_2} + \partial_2 f(\theta \phi_{c_2}, \theta \psi_{c_2}) \psi_{c_2} \right] d\theta
\leq \int_0^1 \left\{ \left[ \partial_1 f(0, 0) - \partial_1 f(\theta \phi_{c_2}, \theta \psi_{c_2}) \right] \phi_{c_2} + \left[ \partial_2 f(0, 0) - \partial_2 f(\theta \phi_{c_2}, \theta \psi_{c_2}) \right] \psi_{c_2} \right\} d\theta
\leq L \left[ \phi_{c_2}(-x + p_2(t)) + \psi_{c_2}(-x + p_2(t)) \right]^2.
\]

Thus, we have

\[
G_1(x, t) \leq L \left[ \phi_{c_2}(-x + p_2(t)) + \psi_{c_2}(-x + p_2(t)) \right]^2 \text{ for } (x, t) \in \mathbb{R} \times (-\infty, 0].
\]

Similarly, we can show that

\[
G_1(x, t) \leq L \left[ \phi_{c_1}(x + p_1(t)) + \psi_{c_1}(x + p_1(t)) \right]^2 \text{ for } (x, t) \in \mathbb{R} \times (-\infty, 0].
\]

Then, the claim follows.

Next we divide \(S_2\) into two subsets: \((-\infty, 0]\) and \([0, +\infty)\) to estimate \(H_1.\)

**Subcase (i).** \(x \in [0, +\infty).\) Note that \(p_2(t) \leq p_1(t) \leq 0\) for \(t \leq 0.\) By (34)–(36), we
have
\[
H_1(x, t) \leq L \frac{\phi_{c_2}(-x + p_2(t)) + \psi_{c_2}(-x + p_2(t))}{\phi'_{c_2}(-x + p_2(t))} [\phi_{c_2}(-x + p_2(t)) + \psi_{c_2}(-x + p_2(t))]
\leq L \left[ \frac{1}{\mu(c_2)} + \frac{\psi_{c_2}(-x + p_2(t))}{\phi_{c_2}(-x + p_2(t))} \right] [1 + \eta(c_2)] K(c_2) e^{\lambda_1(c_2) p_2(t)}
\leq L \left[ \frac{1}{\mu(c_2)} + \frac{\eta(c_2)}{\mu(c_2)} \right] [1 + \eta(c_2)] K(c_2) e^{\lambda_1(c_2) p_2(t)}
\leq L K(c_2) \left[ \frac{1}{\mu(c_2)} + \eta(c_2) \right] e^{\lambda_1(c_2) p_1(t)}
\]

Subcase (ii). \( x \in (-\infty, 0] \). Noting that \( \lambda_1(c_2) < \lambda_1(c_1) \), by (34)–(36), we have
\[
H_1(x, t) \leq L \frac{\phi_{c_1}(x + p_1(t)) + \psi_{c_1}(x + p_1(t))}{\phi'_{c_1}(x + p_1(t))} [\phi_{c_1}(x + p_1(t)) + \psi_{c_1}(x + p_1(t))]
\leq L K(c_1) \left[ \frac{1}{\mu(c_1)} + 2 \eta(c_1) \right] e^{\lambda_1(c_1) p_1(t)} \leq L K(c_1) \left[ \frac{1}{\mu(c_1)} + \eta(c_1) \right] e^{\lambda_1(c_2) p_1(t)}
\]

Therefore, \( L_1[\overline{\pi}](x, t) \geq 0 \) for \( (x, t) \in S_1 \cup S_2 \).

Step 2. We now verify that \( L_2[\overline{\pi}](x, t) \geq 0 \) for \( (x, t) \in S_3 \cup S_4 \).

Case (I). For \( (x, t) \in S_1 \), \( \overline{\pi}(x, t) \leq K_1 \) and \( \overline{\pi}(x, t) = K_2 \). Then, we have
\[
L_2[\overline{\pi}](x, t) = \beta K_2 - g(\overline{\pi}(x, t)) \geq \beta K_2 - g(K_1) = 0.
\]

Case (II). For \( (x, t) \in S_1 \), \( \overline{\pi}(x, t) \leq \phi_{c_1}(x + p_1(t)) + \phi_{c_2}(-x + p_2(t)) \leq 2K_1 \) and \( \overline{\pi}(x, t) = \psi_{c_1}(x + p_1(t)) + \psi_{c_2}(-x + p_2(t)) \). Let \( \sigma = 2 \). We make the extension for the function \( g \) on \([0, 2K_1] \). Thus, \( g \in C^1([0, 2K_1]) \), \( g'(u) \geq 0 \) for \( u \in [0, 2K_1] \), and
\[
L_2[\overline{\pi}](x, t) = \frac{G_2(x, t)}{\psi'_{c_1}(x + p_1(t)) + \psi'_{c_2}(-x + p_2(t))},
\]
and
\[
G_2(x, t) = g(\phi_{c_1} + \phi_{c_2}) - g(\phi_{c_1}) - g(\phi_{c_2}).
\]

In view of \( g'(u) \leq g'(0) \) for all \( u \in [0, 2K_1] \) and \( |g'(u_1) - g'(u_2)| \leq L_2|u_1 - u_2| \) for all \( u_1, u_2 \in [0, 2K_1] \), it is easy to prove that \( G_2(x, t) \leq L_2 \phi_{c_1}^2((-1)^{i+1} x + p_i(t)), \ i = 1, 2 \).

We also consider two subcases: \( (-\infty, 0] \) and \([0, +\infty) \) to estimate \( H_2 \).

Subcase (i). \( x \in (-\infty, 0] \). It follows from (34)–(36) that
\[
H_2(x, t) \leq \frac{L_2 \phi_{c_1}^2(x + p_1(t))}{\psi'_{c_1}(x + p_1(t)) + \psi'_{c_2}(-x + p_2(t))} \leq \frac{L_2 K(c_1)}{\mu^2(c_1)} e^{\lambda_1(c_1) (x + p_1(t))} \leq \frac{L_2 K(c_1)}{\mu^2(c_1)} e^{\lambda_1(c_2) p_1(t)}.
\]
Subcase (ii). $x \in [0, +\infty)$. Similarly, by (34)–(36), we have
\[
H_2(x, t) \leq \frac{L_2\phi_c(-x + p_2(t))}{\psi_c'(-x + p_2(t))/\phi_c(-x + p_2(t))} \leq \frac{L_2K(c_2)}{\mu^2(c_2)} e^{\lambda_1(c_2)(-x + p_2(t))} \leq \frac{L_2K(c_2)}{\mu^2(c_2)} e^{\lambda_1(c_2)p_1(t)}.
\]

Hence, $L_2[\overline{w}(x, t) \geq 0$ for $(x, t) \in S_3 \cup S_4$.

**Step 3.** We now show that there exists $T < 0$ such that $\overline{w}(x, t)$ is a supersolution of (3) on $(-\infty, T)$. We first show the following claim.

**Claim.** There exists $T_1 < 0$ such that for every $t < T_1$, there are only a finite number of points in $x \in \mathbb{R}$ so that $\phi_{c_1}(x + p_1(t)) + \phi_{c_2}(-x + p_2(t)) = K_1$.

We only consider the case $\lambda_1(c_2) \geq -\lambda_1(c_1)$ and $\lambda_1(c_1) \geq -\lambda_3(c_2)$, the other cases can be discussed similarly. Obviously, there exists $T_1 < 0$ such that $k(c_1)e^{\lambda_1(c_1)p_1(t)} - K(c_2)e^{\lambda_1(c_2)p_2(t)} > 0$ and $k(c_2)e^{\lambda_3(c_2)p_2(t)} - K(c_1)e^{\lambda_1(c_1)p_1(t)} > 0$ for any $t < T_1$. Fix $t < T_1$. Then, by (34)–(37), for sufficiently large $x > -p_1(t)$,
\[
\begin{align*}
\phi_{c_1}(x + p_1(t)) + \phi_{c_2}(-x + p_2(t)) & \leq K_1 - k(c_1)e^{\lambda_1(c_1)(x + p_1(t))} + K(c_2)e^{\lambda_1(c_2)(-x + p_2(t))} \\
& \leq K_1 - e^{-\lambda_1(c_2)x} \left[k(c_1)e^{\lambda_3(c_1)p_1(t)} - K(c_2)e^{\lambda_1(c_2)p_2(t)}\right] < K_1.
\end{align*}
\]

Then, for sufficiently large $|x|$ with $x < 0$.
\[
\begin{align*}
\phi_{c_1}(x + p_1(t)) + \phi_{c_2}(-x + p_2(t)) & \leq K_1 - k(c_2)e^{\lambda_3(c_2)(-x + p_2(t))} + K(c_1)e^{\lambda_1(c_1)(x + p_1(t))} \\
& \leq K_1 - e^{\lambda_1(c_1)x} \left[k(c_2)e^{\lambda_3(c_2)p_2(t)} - K(c_1)e^{\lambda_1(c_1)p_1(t)}\right] < K_1.
\end{align*}
\]

Take $T = \min(T_1, T_2)$. Fix any $s < T$. Note that if $x(t_0) \in \mathbb{R}$ such that $\phi_{c_1}(x(t_0) + p_1(t_0)) + \phi_{c_2}(-x(t_0) + p_2(t_0)) = K_1$ for $s \leq t_0 < T$, then it is easy to see that $\frac{\partial}{\partial x}\overline{w}(x(t_0) + 0, t_0) \leq \frac{\partial}{\partial x}\overline{w}(x(t_0) + 0, t_0)$. Using a similar method as in [27, Lemma 3.6], we can show that for any $x \in \mathbb{R}$ and $s \leq \tau < t < T$,
\[
\overline{w}(x, t) \geq T_1(t - \tau)\overline{w}(x, \tau) + \int_{\tau}^{t} T_1(t - r) f(\overline{w}(x, r), \overline{w}(x, r)) dr,
\]
and
\[
\overline{u}(x, t) \geq T_2(t - \tau)\overline{u}(x, \tau) + \int_{\tau}^{t} T_2(t - r) g(\overline{u}(x, r)) dr.
\]

Therefore, $\overline{w}(x, t)$ is a supersolution of (3) on $(-\infty, T)$.

**Proof of Theorem 3.** For $n \in \mathbb{N}$, we denote
\[
\varphi^n(x) := (\varphi^n_1(x), \varphi^n_2(x)) = \max \left\{ \Phi_{c_1}(x - c_1n + \omega_1), \Phi_{c_2}(-x - c_2n + \omega_2) \right\}, \ x \in \mathbb{R}.
\]

Consider the following initial value problem of (3):
\[
\begin{align*}
u_2(x, t) = du_{xx}(x, t) + f(u(x, t), v(x, t)), \ x \in \mathbb{R}, t > -n, \\
v_1(x, t) = -\beta v(x, t) + g(u(x, t)), \ x \in \mathbb{R}, t > -n, \\
(u(x, -n), v(x, -n)) = \varphi^n(x), \ x \in \mathbb{R}.
\end{align*}
\]

(43)
We first show the following claim.

**Claim.** \( \varphi^n \in [0, K]_X \) and there exists \( L' > 0 \), which is independent of \( n \), such that

\[
\sup_{x \in \mathbb{R}} |\varphi^n(x + \eta) - \varphi^n(x)| \leq L' \eta, \quad \forall \eta > 0. \tag{44}
\]

Clearly, \( 0 \leq \varphi^n(\cdot) \leq K \). Note that \( 0 \leq \psi_c'(t) \leq \frac{1}{\varepsilon} |\beta K_2 + g(K_1)| : = L_c, \; \forall t \in \mathbb{R}, \) and for each \( n \in \mathbb{N} \), there exists \( x_n \in \mathbb{R} \) such that

\[
\varphi^n_2(x) = \begin{cases} 
\psi_{c_1}(x - c_1 n + \omega), & x > x_n, \\
\psi_{c_1}(x_n - c_1 n + \omega) = \psi_{c_2}(-x_n - c_2 n + \omega_2), & x = x_n, \\
\psi_{c_2}(-x - c_2 n + \omega_2), & x < x_n.
\end{cases}
\]

For any \( x \in \mathbb{R} \) and \( \eta > 0 \), if \( x + \eta > x_n \), then

\[
|\varphi^n_2(x + \eta) - \varphi^n_2(x)| = |\psi_{c_1}(x + \eta - c_1 n + \omega) - \psi_{c_1}(x - c_1 n + \omega)| \leq L_c \eta, \tag{45}
\]

and if \( x < x + \eta \leq x_n \), then

\[
|\varphi^n_2(x + \eta) - \varphi^n_2(x)| = |\psi_{c_2}(-x - \eta - c_2 n + \omega_2) - \psi_{c_2}(-x - c_2 n + \omega_2)| \leq L_c \eta. \tag{46}
\]

Moreover, if \( x \leq x_n \leq x + \eta \), then

\[
|\varphi^n_2(x + \eta) - \varphi^n_2(x)| = |\varphi^n_2(x) - \varphi^n_2(x_n)| \leq L_c \eta. \tag{47}
\]

Taking \( L' = \max\{L_{c_1}, L_{c_2}\} \), (44) follows from (45)-(47). In view of

\[
0 \leq \phi_\xi'(\xi) = \frac{1}{d(\lambda_4 - \lambda_3)} \left[ \int_{-\xi}^0 \lambda_3 e^{\lambda_3(s-s)} H(\phi)(s) ds + \int_{\xi}^{+\infty} \lambda_4 e^{\lambda_4(s-s)} H(\phi)(s) ds \right]
\leq \frac{2}{\sqrt{c^2 + 4 L_1 d}} [L_1 K_1 + \max_{(u,v) \in [0,K]} |f(u,v)|] := L', \; \xi \in \mathbb{R},
\]

where \( \lambda_3, \lambda_4, L_1, H(\phi) \) are defined as in the proof of Lemma 3, we can similarly show that for any \( \eta > 0 \),

\[
\sup_{x \in \mathbb{R}} |\varphi^n(\xi) - \varphi^n(x)| \leq \max\{L'_{c_1}, L'_{c_2}\} \eta.
\]

Therefore, we obtain \( \varphi^n \in [0, K]_X \) and the claim follows.

From Lemma 14, (43) has a unique solution \( w^n(x, t) \) which satisfies \( 0 \leq w^n(x, t) \leq K \) for \( (x, t) \in \mathbb{R} \times [-n, +\infty) \) and \( w(x, t) \leq w^{n+1}(x, t) \leq w^n(x, t) \leq w(x, t) \) for \( (x, t) \in \mathbb{R} \times [-n, T] \). Moreover, by Lemma 15 and (44), there exist a function \( W_{c_1,c_2,\omega_1,\omega_2}(x, t) := (u(x, t), v(x, t)) \) and a subsequence \( \{w^{n_k}(x, t)\}_{k \in \mathbb{N}} \) of \( \{w^n(x, t)\}_{n \in \mathbb{N}} \) such that \( w^{n_k}(x, t), w^{n_k}_t(x, t), w^{n_k}_{xx}(x, t) \) converge uniformly in any compact set \( S \subset \mathbb{R}^2 \) to

\[
W_{c_1,c_2,\omega_1,\omega_2}(x, t), \; \frac{\partial}{\partial t} W_{c_1,c_2,\omega_1,\omega_2}(x, t), \; \frac{\partial^2}{\partial x^2} u(x, t).
\]

Since \( w^{n+1}(x, t) \leq w^n(x, t) \) for \( (x, t) \in \mathbb{R} \times [-n, +\infty) \), \( \lim_{n \to +\infty} w^n(x, t) = W_{c_1,c_2,\omega_1,\omega_2}(x, t) \). Clearly, \( W_{c_1,c_2,\omega_1,\omega_2}(x, t) \) is an entire solution of (3) and satisfies

\[
\max \{ \Phi_{c_1}(x + c_1 t + \omega_1), \Phi_{c_2}(-x + c_2 t + \omega_2) \}\]

\[
\leq W_{c_1,c_2,\omega_1,\omega_2}(x, t) \leq \min \{ K, \Phi_{c_1}(x + p_1(t)) + \Phi_{c_2}(-x + p_2(t)) \} \tag{48}
\]

for \( (x, t) \in \mathbb{R} \times (-\infty, T] \). Moreover,

\[
w(x, t) \leq W_{c_1,c_2,\omega_1,\omega_2}(x, t) \leq K \quad \text{for} \; (x, t) \in \mathbb{R}^2. \tag{49}
\]
Thus, \( W_{c_1,c_2,\theta_1,\theta_2}(x,t) \) is also an entire solution of (3). Therefore, to complete the proof, it suffices to show that \( W_{c_1,c_2,\omega_1,\omega_2}(x,t) \) meets Theorem 3 with \( \theta_i = \omega_i \), \( i = 1, 2 \).

Now, we prove the assertion (i). Since \( w^n(x,t) \geq w(x,t) \geq w(x,-n) = w^n(x,-n) \), \( \frac{\partial}{\partial t} w^n(x,t) \geq 0 \) for \( (x,t) \in \mathbb{R} \times (0, +\infty) \) by the order-preserving of the solution semiflow (see Lemma 14). Then, \( \frac{\partial}{\partial t} W_{c_1,c_2,\omega_1,\omega_2}(x,t) \geq 0 \) for all \( (x,t) \in \mathbb{R}^2 \). Note that

\[
\frac{\partial u_t}{\partial t} = d \frac{\partial^2 u_t}{\partial x^2} + \partial_1 f(u,v) u_t + \partial_2 f(v,u) v_t \geq d \frac{\partial^2 u_t}{\partial x^2} + mu_t,
\]

\[
\frac{\partial v_t}{\partial t} = -\beta v_t + g'(u) u_t \geq -\beta v_t,
\]

where \( x \in \mathbb{R} \) and \( m = \min_{(u,v) \in [0,K]} \partial_1 f(u,v) \). For any given \( \tau \in \mathbb{R} \), we have

\[
u_t(x,t) \geq v_t(x,\tau) e^{-\beta(t-\tau)} \geq 0, \quad \forall x \in \mathbb{R}, t > \tau,
\]

where \( K(x,t) = \int_{-\infty}^{+\infty} K(x - y, t - \tau) u_t(y, \tau) dy \geq 0, \quad \forall x \in \mathbb{R}, t > \tau \).

Using (48) and (49), the proofs for (6)–(7), (ii)–(v) and (ix) in Theorem 3 are straightforward and thus omitted.
In view of \( \lim_{n \to +\infty} w^n(x, t) = W_{c_1, c_2, \omega_1, \omega_2}(x, t) \), we get
\[
0 \leq W_{c_1, c_2, \omega_1, \omega_2}(x, t) - \Phi_{c_2}(x + c_1 t + \omega_1) \leq B_{c_2}(1, A_{c_2})e^{\lambda_1(c_2)(-x + c_2 t + \omega_2)}
\]
for all \((x, t) \in \mathbb{R}^2\), which implies that \( W_{c_1, c_2, \omega_1, \omega_2}(x, t) \) converges to \( \Phi_{c_2}(x + c_1 t + \omega_1) \) as \( \omega_2 \to -\infty \) uniformly on \((x, t) \in [N, +\infty) \times (-\infty, a] \) for any \( N, a \in \mathbb{R} \). Similarly, we can show that if \( c_1 > c_{\min} \), then \( W_{c_1, c_2, \omega_1, \omega_2}(x, t) \) converges to \( \Phi_{c_2}(-x + c_2 t + \omega_2) \) as \( \omega_1 \to -\infty \) uniformly on \((x, t) \in (-\infty, N] \times (-\infty, a] \) for any \( N, a \in \mathbb{R} \).

Finally, we show that (vii) and (viii) hold. Note that for any fixed \( x \in \mathbb{R} \) and \( t \ll -1 \),
\[
\max \{ \Phi_{c_1}(x + c_1 t + \omega_1), \Phi_{c_2}(-x + c_2 t + \omega_2) \} \leq W_{c_1, c_2, \omega_1, \omega_2}(x, t)
\]
\[
\leq \Phi_{c_1}(x + p_1(t)) + \Phi_{c_2}(-x + p_2(t)).
\]
It then follows from Theorem 5 and \( \frac{d}{dx}(c \lambda_1(c)) < 0 \) for \( c > 0 \) that (vii) holds. Using the assertion (vii) and Lemmas 1 and 3, the proof of (viii) is similar to that of [16, Theorem 1.1(vii)] and omitted.

This completes the proof of Theorem 3.

4.3. Proof of Theorem 4.

Lemma 19. The function \( w^-(x, t) = (u^-(x, t), v^-(x, t)) \) defined by
\[
w^-(x, t) = \max \left\{ \chi_1 \Phi_{c_1}(x + c_1 t + \omega_1), \chi_2 \Phi_{c_2}(-x + c_2 t + \omega_2), \Gamma(t + \theta_3) \right\}
\]
is a subsolution of (3) on \((-\infty, +\infty)\), where \( \chi_1, \chi_2 \in \{0, 1\} \) with \( \chi_1 + \chi_2 \geq 1 \), \( \omega_1, \omega_2 \) are defined by (40) and \( \theta_3 \in \mathbb{R} \).

The proof is similar to that of Lemma 17, see also [16, Lemma 3.6]. We omit it here.

Lemma 20. There exists \( T < 0 \) such that the function \( w^+(x, t) = (u^+(x, t), v^+(x, t)) \) defined by
\[
w^+(x, t) = \min \left\{ \chi_1 \Phi_{c_1}(x + p_1(t)) + \chi_2 \Phi_{c_2}(-x + p_2(t)) + (1, b_\ast)e^{\lambda^*(t + \theta_3)}, K \right\}
\]
is a supersolution of (3) on \((-\infty, T]\), where \( \chi_1, \chi_2 \in \{0, 1\} \) with \( \chi_1 + \chi_2 \geq 1 \), \( \theta_3 \in \mathbb{R} \), \( \lambda^* \) is the same as in Lemmas 11, and \( N \) and \( \alpha \) in (39) are defined as in Lemma 18.

Proof. We only consider the case \( \chi_1 = \chi_2 = 1 \). Denote \( \rho(t) = (\rho_1(t), \rho_2(t)) = (1, b_\ast)e^{\lambda^*(t + \theta_3)} \), then \( \rho(t) \) satisfies
\[
\begin{align*}
\rho_1'(t) &= \theta_1 f(0, 0) \rho_1(t) + \theta_2 f(0, 0) \rho_2(t), \\
\rho_2'(t) &= -\beta \rho_2(t) + g'(0) \rho_1(t).
\end{align*}
\]
Define
\[
P_1 = \left\{(x, t) \in \mathbb{R} \times (-\infty, 0) \mid \phi_{c_1}(x + p_1(t)) + \phi_{c_2}(-x + p_2(t)) + \rho_1(t) > K_1 \right\},
\]
\[
P_2 = \left\{(x, t) \in \mathbb{R} \times (-\infty, 0) \mid \phi_{c_1}(x + p_1(t)) + \phi_{c_2}(-x + p_2(t)) + \rho_1(t) < K_1 \right\},
\]
\[
P_3 = \left\{(x, t) \in \mathbb{R} \times (-\infty, 0) \mid \psi_{c_1}(x + p_1(t)) + \psi_{c_2}(-x + p_2(t)) + \rho_2(t) > K_2 \right\},
\]
\[
P_4 = \left\{(x, t) \in \mathbb{R} \times (-\infty, 0) \mid \psi_{c_1}(x + p_1(t)) + \psi_{c_2}(-x + p_2(t)) + \rho_2(t) < K_2 \right\}.
\]
The remainder of the proof is divided into three steps.

Step 1. We first verify that \( L_1[w^+](x, t) \geq 0 \) for \( (x, t) \in P_1 \cup P_2 \).

Case (I). For \( (x, t) \in P_1 \), \( u^+(x, t) = K_1 \) and \( v^+(x, t) \leq K_2 \). Clearly, \( L_1[w^+](x, t) \geq 0 \).
Case (II). For \((x, t) \in P_2\), \(u^+(x, t) = \phi_{c_1}(x + p_1(t)) + \phi_{c_2}(-x + p_2(t)) + \rho_1(t)\). Then,
\[
\mathbb{L}_1[w^+](x, t) = P_1 \phi_{c_1} + P_2 \phi_{c_2} + \rho_1 - d\phi_{c_1} - d\phi_{c_2} - f(\phi_{c_1} + \phi_{c_2} + \rho_1, v^+).
\]
\[
= \left[ [\phi_{c_1} + \phi_{c_2}] N exp(t) - f(\phi_{c_1} + \phi_{c_2} + \rho_1, v^+) \right] + f(\phi_{c_1}, \psi_{c_1}) + f(\phi_{c_2}, \psi_{c_2}) + \rho_1' \nabla \phi_{c_1} \nabla \phi_{c_2} - G_3(x, t),
\]
where \(G_3(x, t) = f(\phi_{c_1} + \phi_{c_2} + \rho_1, v^+) - f(\phi_{c_2}, \psi_{c_2})\).

We claim that \(G_3(x, t) \leq G_1(x, t)\) for all \((x, t) \in P_2\), where \(G_1(x, t)\) given by \((41)\). Indeed, if \(v^+(x, t) = K_2\), then by the condition \((A_3)'\), we have
\[
G_3(x, t) = G_1(x, t) + f(\phi_{c_1} + \phi_{c_2} + \rho_1, K_2) - f(\phi_{c_1} + \phi_{c_2} + \rho_1, v^+) \leq G_1(x, t) + \partial_1 f(0, 0) \rho_1 + \partial_2 f(0, 0) \rho_2 - \rho_1' = G_1(x, t),
\]
by considering two cases: \(\min\{\psi_{c_1} + \psi_{c_2}, K_2\} = \psi_{c_1} + \psi_{c_2}\) and \(\min\{\psi_{c_1} + \psi_{c_2}, K_2\} = K_2\). If \(v^+(x, t) = \psi_{c_1}(x + p_1(t)) + \psi_{c_2}(-x + p_2(t)) + \rho_2(t) \leq K_2\), then we have
\[
G_3(x, t) = G_1(x, t) + f(\phi_{c_1} + \phi_{c_2} + \rho_1, \psi_{c_1} + \psi_{c_2} + \rho_2) - f(\phi_{c_1} + \phi_{c_2}, \psi_{c_1} + \psi_{c_2}) - \rho_1' = G_1(x, t) + \int_0^1 \partial_1 f(\phi_{c_1} + \phi_{c_2} + \theta_1 \rho_1, \psi_{c_1} + \psi_{c_2} + \theta_2 \rho_2) \rho_1 d\theta + \int_0^1 \partial_2 f(\phi_{c_1} + \phi_{c_2} + \theta_1 \rho_1, \psi_{c_1} + \psi_{c_2} + \theta_2 \rho_2) \rho_2 d\theta - \rho_1' \leq G_1(x, t).
\]
Then, for all \((x, t) \in P_2\),
\[
\mathbb{L}_1[w^+](x, t) \geq \left[ \phi_{c_1}' + \phi_{c_2}' \right] N exp(t) - G_1(x, t).
\]
Noting that \(P_2 \subseteq S_2\), by a similar argument to that in Lemma 18, we obtain \(\mathbb{L}_1[w^+](x, t) \geq 0\) for \((x, t) \in P_2\). Therefore, \(\mathbb{L}_1[w^+](x, t) \geq 0\) for all \((x, t) \in P_1 \cup P_2\).

Step 2. We now verify that \(\mathbb{L}_2[w^+](x, t) \geq 0\) for \((x, t) \in P_3 \cup P_4\).

Case (I). For \((x, t) \in P_3\), \(u^+(x, t) \leq K_1\) and \(v^+(x, t) = K_2\). Then, \(\mathbb{L}_2[w^+](x, t) \geq 0\).

Case (II). For \((x, t) \in P_4\), \(u^+(x, t) \leq \phi_{c_1}(x + p_1(t)) + \phi_{c_2}(-x + p_2(t)) + \rho_1(t) \leq 2K_1 + e^{\lambda_1 t}\) and \(v^+(x, t) = \psi_{c_1}(x + p_1(t)) + \psi_{c_2}(-x + p_2(t)) + \rho_2(t)\). Choose \(\sigma_0 > 1\) such that \(2K_1 + e^{\lambda_1 t} \leq \sigma_0 K_1\). We make the extension for the function \(g\) on \([0, \sigma_0 K_1]\). Then \(g'(u) \geq 0\) and \(g'(u) \leq g'(0)\) for all \(u \in [0, \sigma_0 K_1]\). Thus, we obtain
\[
\mathbb{L}_2[w^+](x, t) = \phi_{c_1}' + \phi_{c_2}' + \rho_1' - d\phi_{c_1} - d\phi_{c_2} - g(u^+)
\]
\[
\geq \left( \phi_{c_1}' + \phi_{c_2}' + \rho_1' - d\phi_{c_1} - d\phi_{c_2} - g(u^+) \right) + g'(0) \rho_1 - g(\phi_{c_1} + \phi_{c_2} + \rho_1) + f(\phi_{c_1} + \phi_{c_2} + \rho_1, v^+)
\]
\[
= \left[ \phi_{c_1}' + \phi_{c_2}' \right] N exp(t) - G_2(x, t),
\]
where \(\theta \in (0, 1)\) and \(G_2(x, t)\) given by \((42)\). Since \(P_4 \subseteq S_4\), using a similar argument as in Lemma 18, we get \(\mathbb{L}_2[w^+](x, t) \geq 0\) for \((x, t) \in P_4\). Therefore, \(\mathbb{L}_2[w^+](x, t) \geq 0\) for \((x, t) \in P_3 \cup P_4\).

Step 3. By a similar argument as Lemma 18, we can prove that there exists \(T < 0\) such that \(w^+(x, t)\) is a supersolution of \((3)\) on \((-\infty, T)\). The proof is complete.
Proof of Theorem 4. The proof of Theorem 4 is similar to that of Theorem 3 according to Lemmas 19 and 20, here we omit it.

5. Applications.
Example 5.1. Consider the model (1). Mathematically, we can rescale (1) and only study the rescaled system
\[ \begin{align*}
    u_t(x,t) &= d u_{xx}(x,t) - u(x,t) + \alpha v(x,t), \\
    v_t(x,t) &= -\beta v(x,t) + g(u(x,t)),
\end{align*} \tag{50} \]
where \( \alpha = a_{12}/a_{11} > 0 \) and \( \beta = a_{22}/a_{11} > 0 \). We make the following assumption:

(B): \( g \in C^2([0, K_1], \mathbb{R}), g(0) = g(K_1) - \beta K_1 = 0, g(u) > \frac{\beta}{a} u \) for \( u \in (0, K_1) \),
\[ \alpha g'(K_1) \leq \beta, \quad g(u) \leq g'(0)u \text{ and } g'(u) > 0 \text{ for } u \in [0, K_1], \]
where \( K_1 > 0 \) is a constant.

Let \( F(u,v) = (-u + \alpha v, -\beta v + g(u)) \). If the assumption (B) holds, then \((A_1)-(A_3)\) hold, and hence the conclusions of Theorems 1 and 2 are valid for the system (50). If, in addition, \( g'(u) \leq g'(0) \) for \( u \in [0, K_1] \), then \((A_1), (A_2)\) and \((A_3)'\) hold. Thus, the conclusions of Theorems 3-4 hold true for (50). We remark that the function \( g(u) = \frac{2\beta u}{a(1+u)} \) satisfies the condition \((B)\) with \( K_1 = \frac{1}{2} \).

Example 5.2. We now consider the model (2). As mentioned in the introduction, Zhang and Zhao [30] considered the asymptotic behavior of solutions of (2), and Zhang and Li [31] further considered the monotonicity and uniqueness of traveling wave solutions of (2). However, there has been no results on the entire solutions for the system.

Let \( F(u,v) = (f_1(u) - \gamma_1 u + \gamma_2 v, \gamma_1 u - \gamma_2 v) \). If the reproduction function \( f_1 \) satisfies the following assumption

(C): \( f_1 \in C^2([0, K_1], \mathbb{R}), f_1(0) = f_1(K_1) = 0, f_1'(K_1) < 0, f_1(u) > 0 \) for \( u \in (0, K_1) \), and \( f_1(u) \leq f_1(0) \) for \( u \in [0, K_1] \), where \( K_1 \) is a positive constant, then \((A_1), (A_2)\) and \((A_3)'\) hold. Hence, the conclusions of Theorems 1-4 hold true for (2). Obviously, our results extend and complement those established by [30,31].

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