A Comprehensive Elliptic Integral Solution to the Large Deflection Problems of Thin Beams in Compliant Mechanisms

The elliptic integral solution is often considered to be the most accurate method for analyzing large deflections of thin beams in compliant mechanisms. In this paper, a comprehensive solution based on the elliptic integrals is proposed for solving large deflection problems. By explicitly incorporating the number of inflection points and the sign of the end-moment load in the derivation, the comprehensive solution is capable of solving large deflections of thin beams with multiple inflection points and subject to any kinds of load cases. The comprehensive solution also extends the elliptic integral solutions to be suitable for any beam end angle. Deflected configurations of complex modes solved by the comprehensive solution are presented and discussed. The use of the comprehensive solution in analyzing compliant mechanisms is also demonstrated by examples.

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1 Introduction

Compliant mechanisms, which achieve at least some of their mobility from the deflection of flexible segments rather than from articulated joints only, offer many advantages such as energy storage, increased precision, and reduced wear, backlash and part number [1]. However, the nonlinearity associated with the large deflection problem often complicates the design and analysis of compliant mechanisms. One of the major difficulties lies in accurately modeling the large deflections in compliant mechanisms.

Several methods are currently available for solving the large deflection problems in compliant mechanisms, e.g., the circle-arc method [2], the finite element method, the chain algorithm [1,3–5], the Adomian decomposition method [6], the elliptic integral solution [7], and various pseudo-rigid-body model (PRBM) methods [8–12]. Among these methods, the elliptic integral solution is often considered to be the most accurate method for modeling large deflections of beams that are so thin and flexible that the effects of axial elongation and shear are negligible. Bisshopp and Drucker derived an elliptic integral solution for beams subject to vertical forces [13]. Howell and Lyon et al. [1,9,10,14] presented the elliptic integral solutions for a few load cases where no inflection point is produced in a beam. An elliptic integral solution for the large deflection with an inflection point was derived by Kimball and Tsai [15]. These solutions have the limitation that the slope \( \theta \) of the deflected beam is in the range of \( -\pi + \phi \leq \theta < \phi \) (\( \phi \) denotes the direction of the end force). Shoup presented the solutions to the undulating and nodal elastica based on the elliptic integrals [16,17]. Chen and Zhang [18] derived the elliptic integral solution for static energy in large-deflection beams and used the solution to evaluate the accuracy of PRBM. The elliptic integrals have also been used to solve the fixed-guided problem in which two inflection points might occur [19–21].

In this paper, a comprehensive elliptic integral solution is proposed for solving the large deflections of beams of any end angle and with multiple inflection points. By incorporating the number of inflection points (n) and the sign of the end-moment load (SM) in the derivation, the comprehensive solution is capable of locating all the possible deflected configurations of the beam for a given tip load or tip deflection. The comprehensive solution also extends the elliptic integral solutions to be suitable for any beam end angle. The relationship between the deflection angles at inflection points is revealed for the first time. The comprehensive solution also encompasses the undulating and nodal elastica solutions.

The rest of the paper is organized as follows. In Sec. 2, the basic equations of the large deflection problem are reviewed and the angles at inflection points are further discussed. Section 3 presents the comprehensive elliptic integral solution and discusses a few deflected configurations of complex modes. Two examples are presented to demonstrate the use of the comprehensive solution in Sec. 4. Section 5 presents two case studies to show the capabilities of the comprehensive solution in solving problems with the slope \( \theta \) out of the range of \( -\pi + \phi \leq \theta < \phi \) and with multiple inflection points. The last section has concluding remarks.

2 Large Deflection Beam Equations

Figure 1 shows an initially straight beam subject to an end force \( \eta P \) and an end moment \( M_0 \). The end force can be divided into a vertical component \( P \) and a horizontal component \( \eta P \) and we have

\[
\eta = \sqrt{1 + \eta^2}
\]

Without loss of generality, we assume \( P \) is always positive, while \( S_M \) is introduced to denote the sign of \( M_0 \) as

\[
S_M = \begin{cases} 
1 & M_0 \geq 0 \\
-1 & M_0 < 0 
\end{cases}
\]
Integrating Eq. (5) results in

\[ \frac{d\theta}{ds} = \frac{d\theta}{dx} = \frac{d\theta}{dy} = \frac{d\theta}{dz} = \frac{d\theta}{ds} \]

Note that

\[ \frac{d\theta}{ds} = \frac{d\theta}{dx} = \frac{d\theta}{dy} = \frac{d\theta}{dz} \]

substituting Eq. (10) into Eq. (5), separating variables and integrating yields

\[ \begin{align*}
  a &= \pm \frac{1}{2\sqrt{2}} \int_{0}^{\theta_0} \sqrt{-\sin \theta + n \cos \theta + \lambda} \, d\theta \\
  b &= \pm \frac{1}{2\sqrt{2}} \int_{0}^{\theta_0} \sqrt{-\sin \theta + n \cos \theta + \lambda} \, d\theta
\end{align*} \]

where \(a/L\) and \(b/L\) are the nondimensional coordinates of the tip point along the \(x\)- and \(y\)-axes, respectively. The signs of Eqs. (9) and (11) are chosen in the same way as for Eq. (4). Equations (9) and (11) together serve as the general equations for the large deflection problem.

The elastic theory [16] shows that a deflected beam may possess an arbitrary number of inflection points. Let \(m (m > 0)\) denote the number of inflection points. Knowing that \(M = 0\) at inflection points, Eqs. (5) and (6) are reduced to the following form at inflection points

\[ \lambda = \sin \theta - n \cos \theta \]

Equation (12) has an infinite set of solutions for \(\theta\) and each of the solutions (denoted as \(\theta\)) represents the deflection angle at an inflection point, as shown in Fig. 2. The solution is given as

\[ \hat{\theta}_j = 2k\pi + \phi \pm \cos^{-1}(\lambda/\eta) \quad k = 0, \pm 1, \pm 2 \ldots \]  

where \(j\) indicates the \(j\)th inflection point numbering from the fixed end of the deflected beam and \(\phi\) represents the angle of the force applied at the free end (as marked in Fig. 1).

\[ \phi = \pi/2 + \tan^{-1} n \]

Equation (13) also reveals the relationship between the deflection angles at inflection points.

The deflection curves can be divided into two groups according to the sign of \(M_i\) (the resulting moment at the fixed end of the beam, as marked in Fig. 1). Figure 3(a) presents the deflected configurations with positive curvature at the fixed ends, i.e., \(M_i > 0\), and examples of \(m = 1, 2, 3\). Because the curvature changes sign at inflection points, we know \(S_i = (-1)^m S_M = 1\), where \(S_i\) denotes the sign of \(M_i\).
and

\[ a = \frac{S}{2\sqrt{2}} \sum_{j=0}^{m} (-1)^{j} \int_{\theta_{j}}^{0} \cos \theta d\theta \]

\[ b = \frac{S}{2\sqrt{2}} \sum_{j=0}^{m} (-1)^{j} \int_{\theta_{j}}^{0} \sin \theta d\theta \]

Without loss of generality, we define \( \hat{\theta}_{0} = 0 \) and \( \hat{\theta}_{m+1} = \theta_{0} \). Equation (16) will be solved using the elliptic integrals in Sec. 3.

### 3 Comprehensive Elliptic Integral Solution

In this section, the elliptic integrals are used to solve Eq. (16). The incomplete elliptic integrals of the first and second kinds are defined as [22]

\[ F(\gamma, t) = \int_{0}^{t} d\delta \sqrt{1 - t^2 \sin^2 \delta} \]

and

\[ E(\gamma, t) = \int_{0}^{t} \sqrt{1 - t^2 \sin^2 \delta} d\delta \]

respectively, where \( \gamma \) is called the amplitude and \( t \) \((-1 \leq t \leq 1)\) the modulus. When \( \gamma = \pi/2 \), they become the complete elliptic integrals of the first and second kinds and are denoted as \( F(\gamma) \) and \( E(\gamma) \), respectively.

Due to the limitation of the range of elliptic integrals, the elliptic integral solutions to Eq. (16) are divided into two parts: \(|\hat{\gamma}| > \eta\) and \(|\hat{\gamma}| \leq \eta\).

#### 3.1 Case I: \(|\hat{\gamma}| > \eta\). When \(|\hat{\gamma}| > \eta\), the deformed curve changes monotonously [15] and there is therefore no inflection point \((m = 0)\). Equation (16) can be written as

\[ x = \frac{S}{\sqrt{2}} \sum_{j=0}^{m} (-1)^{j+1} \int_{\theta_{j}}^{0} \frac{d\theta}{\sqrt{\sin \theta + n \cos \theta + \hat{\gamma}}} \]

\[ a = \frac{S}{2\sqrt{2}} \sum_{j=0}^{m} (-1)^{j+1} \cos \theta d\theta \]

\[ b = \frac{S}{2\sqrt{2}} \sum_{j=0}^{m} (-1)^{j+1} \sin \theta d\theta \]

where

\[ f = F(\gamma_{2}, t) - F(\gamma_{1}, t) \]

\[ e = E(\gamma_{2}, t) - E(\gamma_{1}, t) \]

\[ c = \sqrt{\lambda + n} - \sqrt{\hat{\lambda} - \sin \theta_{o} + n \cos \theta_{o}} \]

\[ t = \sqrt{\frac{2\eta}{\lambda + n}} \]

\[ \gamma_{1} = \sin^{-1} \frac{\eta - n}{2\eta} \]

\[ \gamma_{2} = -[0.5k\pi + (-1)^{k+1} \sin^{-1} \frac{\eta + \sin \theta_{o} - n \cos \theta_{o}}{2\eta}] \]

for \((k-1)\pi + \phi < \theta_{o} \leq k\pi + \phi \) \((k = 0, \pm 1, \pm 2 \cdots)\)

and \([-0.5k\pi\) gives the largest integer less than or equal to \(-0.5k\).
3.2 Case II: $|\lambda| \leq \eta$. For $|\lambda| \leq \eta$, the deformed curve may have an arbitrary number of inflection points ($m \geq 0$). Equation (16) can be expressed using the elliptic integrals as

$$
\begin{align*}
\alpha &= \frac{S_r f}{\sqrt{\eta}} \\
\frac{a}{L} &= \frac{S_r}{2\pi^{3/2}} \left[ -mf + 2\eta e + \sqrt{2\eta}c \right] \quad \text{for } |\lambda| \leq \eta \\
\frac{b}{L} &= \frac{S_r}{2\pi^{3/2}} \left[ mf - 2ne + n\sqrt{2\eta} \right]
\end{align*}
$$

(20)

where

$$
\begin{align*}
f &= \left\{ \begin{array}{ll}
(-1)^m F(\gamma_2, t) - F(\gamma_1, t) - 2 \sum_{j=1}^{m} (-1)^j F(\gamma_j, t) & m \geq 1 \\
F(\gamma_2, t) - F(\gamma_1, t) & m = 0
\end{array} \right.
\end{align*}
$$

$$
\begin{align*}
e &= \left\{ \begin{array}{ll}
(-1)^m E(\gamma_2, t) - E(\gamma_1, t) - 2 \sum_{j=1}^{m} (-1)^j E(\gamma_j, t) & m \geq 1 \\
E(\gamma_2, t) - E(\gamma_1, t) & m = 0
\end{array} \right.
\end{align*}
$$

$$
\begin{align*}
c &= \sqrt{\lambda + n - (-1)^m \sqrt{\lambda - \sin \theta_o + n\cos \theta_o}}
\end{align*}
$$

$$
\begin{align*}
t &= \sqrt{\frac{\lambda + \eta}{2\eta}}
\end{align*}
$$

$$
\begin{align*}
S_r &= (-1)^m S_M \\
\gamma_1 &= \sin^{-1} \sqrt{\frac{\eta - n}{\lambda + \eta}}
\end{align*}
$$

$$
\begin{align*}
\gamma_2 &= \sin^{-1} \sqrt{\frac{\eta + n}{\lambda + \eta}}
\end{align*}
$$

(21)

For $(k - 1)\pi + \phi < \theta_o \leq k\pi + \phi (k = 0, \pm 1, \pm 2 \cdots)$

$$
\begin{align*}
\gamma_j &= \sin^{-1} \sqrt{\frac{\eta + n}{\lambda + \eta}}
\end{align*}
$$

The above derivation is based on the assumption of $P > 0$. In the case that $P < 0$, the signs of $n, \kappa, \theta_o, S_M$ and $b$ (but not $a$) are reversed in Eqs. (19) and (20).

3.3 Discussion of Comprehensive Solution. Generally speaking, a deflected configuration with more inflection points corresponds to a higher strain-energy level in the beam. Unless otherwise constrained, a deflected beam will have the least number of inflection points possible that still satisfies the governing equations and the boundary conditions. Therefore, when solving large deflection problems in compliant mechanisms, an iterative process can be employed to incrementally increase $m$ from 0 to a number that yields a feasible solution (in this work, the build-in function “fsolve” in MATLAB was used with “TolFun” was set to $10^{-8}$). As a starting point, it is reasonable to assume

$$
\begin{align*}
-2\pi + \phi < \theta \leq \phi
\end{align*}
$$

(22)

thus only the following two solutions of $\hat{\theta}_j$ in Eq. (13) need to be considered

$$
\begin{align*}
\hat{\theta}_j &= -2\pi + \phi + \cos^{-1}(\lambda/\eta) \quad \text{or} \quad \hat{\theta}_j = \phi - \cos^{-1}(\lambda/\eta)
\end{align*}
$$

(23)

It is interesting to note that, for the range of Eq. (21), all the angles at odd inflection points are equal, and the angles at even inflection points are also equal. Substituting Eq. (22) into Eq. (20) yields

$$
\begin{align*}
a &= \frac{S_r f}{\sqrt{\eta}} \\
\frac{a}{L} &= \frac{S_r}{2\pi^{3/2}} \left[ -mf + 2\eta e + \sqrt{2\eta}c \right] \quad \text{for } -2\pi + \phi < \theta \leq \phi \\
\frac{b}{L} &= \frac{S_r}{2\pi^{3/2}} \left[ mf - 2ne + n\sqrt{2\eta} \right] \quad \text{and } |\lambda| \leq \eta
\end{align*}
$$

(24)

This indicates that $a/L$ and $b/L$ are independent of $S_M$ and $m$, that is to say, the deflected beam has multiple possible configurations for the same tip deflection, as illustrated in Fig. 5. Even for a given $m$ that is even, there still exist two deflection configurations, with one corresponding to $S_M = 1$ and the other to $S_M = -1$. This special case corresponds to the fixed-guided condition, which is an important case that can be found in many compliant mechanisms, e.g., fully compliant bistable mechanism [20,21] and parallel guided mechanisms.

Now consider a comprehensive solution for a deflection where $\theta$ is beyond the scope outlined in Eq. (21) and is therefore also beyond the range of solutions presented in Refs. [1] and [15], which are valid for $-\pi + \phi < \theta < \phi$. Figure 6 illustrates the deflected curves for $S_M = 1, n = 0$, and $\theta_0 = 3\pi$ obtained by Eq. (19). It is shown that the deflected curve tends to a circle with increasing $\kappa$, which concurs with the results in the nodal
elastica solution [17]. The graphs in Fig. 7 plot the deflections calculated using Eq. (20) for $\kappa = \frac{2}{p} + \frac{\mu}{C_0}$ and $\kappa = \frac{4}{p} + \frac{\mu}{C_0}$.

The comprehensive solution is also useful for pure-force and pure-moment load cases. When the end load is a pure force ($M_o = 0$), $S_M$ can be both $+1$ and $-1$, and Eq. (20) is used to solve for the force index $a$ and the coordinates of the tip point for both cases. When the end load is a pure moment, i.e., $P = 0$, the comprehensive solution can be used to approximate the deflection by using a very large value for $\kappa$ instead of $\infty$, e.g., $10^9$. Figure 8 compares the tip point locus approximated by the comprehensive solution and that of [1] in the range of $h_o \leq \frac{\pi}{2}$. The two loci agree well and the maximum root mean square error is $\frac{\text{error}}{L_{max}} = \sqrt{\left|a/L - (a/L)_{H}\right|^2 + \left|b/L - (b/L)_{H}\right|^2} = 3.52 \times 10^{-4}$ (subscript “H” indicates that the value is obtained using the solution on Page 45 of Ref. [1]).

4 Verification

In this section, two examples taken from the literature are presented to verify the effectiveness of the comprehensive solution.

4.1 Partial Compliant Four-Bar Mechanism. Figure 9 shows a partially compliant four-bar mechanism taken from Ref. [11]. In the mechanism, Link DQ is compliant while links AB and

![Fig. 5 The deflected configurations corresponding to $m = 2$ and $m = 4$ when $\theta = 0$](image)

![Fig. 6 The deflected curves for $S_M = 1$, $n = 0$, and $\theta = 3\pi$](image)

![Fig. 7 The curves of $m = 1$, $S_M = -1$, $\kappa = 0.01$, $n = 10$, and $\theta = \pi$](image)

![Fig. 8 Comparison of the tip locus of the comprehensive solution with that of Ref. [1] for a beam subject to a pure moment](image)

![Fig. 9 A partially compliant four-bar mechanism containing a flexible beam DQ](image)
Applying the static equilibrium for link BQ yields

\[ Q - L_{CQ} \sin \theta_a - L_{BC} \sin(\theta_2 - \theta_a) = L_{AB} \cos \theta_1 \]
\[ Q + L_{CQ} \cos \theta_a - L_{BC} \cos(\theta_2 - \theta_a) = L_{AB} \sin \theta_1 + L_{DA} \]  

(Eq. 25)

The input moment \( T_{in} \) can be obtained by applying moment equilibrium for link AB

\[ T_{in} = P L_{AB} (\cos \theta_1 + n \sin \theta_1) \]  

(Eq. 27)

Figure 10 shows the free-body diagrams for links DQ, AB, and BQ. Applying the static equilibrium for link BQ yields

Further obverse the change of the inflection point, the moments at both ends of DQ (\( M_a \) and \( M_p \)) and the forces applied at Q are plotted as functions of \( \theta_1 \) in Fig. 12. The deflection of Link DQ starts with one inflection point (\( m = 1 \)). The inflection point disappears at \( \theta_1 = 233 \) deg where \( M_D = 0 \), and returns at \( \theta_1 = 265 \) deg where \( M_D = 0 \). The kinetostatic behavior of the mechanism achieved by Eq. (27) is plotted in Fig. 13. \( T_{in} \) equals zero at four positions: \( A (\theta_1 = 0 \) or \( 360 \) deg) and \( C \) are stable equilibrium positions and \( B \) and \( D \) are unstable equilibrium positions. The results obtained using a finite element analysis (FEA) model built with \textsc{ansys} software (Link DQ was meshed into 200 elements with Beam 188) are also shown in Fig. 13 for comparison. It shows that the result of the comprehensive solution agrees well with FEA. On a personal computer with a processor of 2.4 GHz, it takes 25.9 s for the comprehensive solution implemented in \textsc{matlab} to obtain the results, while the FEA model consumes 387.8 s on average to complete the same calculation.

### 4.2 Fixed-Guided Compliant Mechanism

The comprehensive elliptic integral solution was also used to analyze the kinetostatic behavior of a fixed-guided compliant mechanism such as used in many full compliant bistable mechanisms [20,21]. Figure 14 illustrates an initially-straight fixed-guided beam, of which the fixed end has an angle \( \beta \) with the horizontal and the guided end moves vertically. A few authors have contributed to this problem, for example, Zhao et al. [19] presented a numerical method, Holst et al. [21] considered axial deflection in their elliptic integral solution, and Kim [23] proposed a curve decomposition method to simplify the derivation and calculation.

When a vertical force \( F \) is applied at the guided end, the beam is deflected to arrive at a static equilibrium state. During bending, the end slope is constrained to be constant, thus we have \( \theta_a \equiv 0 \). The coordinates of the guided end are given as

\[ a = L - \delta \sin \beta \]
\[ b = -\delta \cos \beta \]  

(Eq. 28)
where \(\delta\) is the displacement of the guided end. \(F_v\) can be solved as

\[
F_v = nP \sin \beta - P \cos \beta
\]

(29)

Given the displacement \(\delta\), the coordinates of the guided end, \(a\) and \(b\), can be solved by Eq. (28). Substituting \(a\) and \(b\) into Eq. (23), \(\kappa\) and \(n\) can be obtained by solving Eq. (24) for both \(SM = 1\) and \(SM = -1\).

The parameters of the beam are the same as those used in Ref. [19]: \(E = 160\) GPa, \(b = 1\) mm, \(h = 0.1\) mm, \(L = 30\) mm, and \(\beta = 30\) deg. Figure 15 shows two possible deflection paths of the beam. In general, both deflection paths start with configurations with two inflection points \((m = 2)\), and end with one of the inflection points having disappeared \((m = 1)\). For the deflection process shown in Fig. 15(a), \(SM = 1\) \((S_1 = 1\) for \(m = 2\) and \(S_1 = -1\) for \(m = 1\)) and the first inflection point moves toward the fixed end of the beam until it disappears. For the deflection process shown in Fig. 15(b), \(S_1 = -1\) \((SM = -1\) for \(m = 2\) and \(SM = 1\) for \(m = 1\)), the second inflection point moves toward the guided end until it disappears. As listed in Table 1, the load-deflection relationships of the two deflection paths are identical except the end moments have opposite signs where two inflection points exist. Reference [21] shows a photograph of a device consisting of two parallel fixed-guided beams, in which one beam is deflected with \(SM = 1\), the other deflected with \(SM = -1\), and both carry two inflection points \((m = 2)\). For the purpose of comparison, we overlay the results of the comprehensive elliptic integral solution on the photograph of Ref. [21] (photo courtesy of Brian Jensen), as shown in Fig. 16, with good agreement.

Figure 17 shows the change of force \(F_v\) versus \(\delta\) for different \(\beta\). Reference [21] shows a photograph of a device consisting of two parallel fixed-guided beams, in which one beam is deflected with \(SM = 1\), the other deflected with \(SM = -1\), and both carry two inflection points \((m = 2)\). For the purpose of comparison, we overlay the results of the comprehensive elliptic integral solution on the photograph of Ref. [21] (photo courtesy of Brian Jensen), as shown in Fig. 16, with good agreement.

### Table 1 The end forces and moments for the two deflection paths

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<th>(\delta)</th>
<th>(P) (N)</th>
<th>(n)</th>
<th>(M_o) (Nm)</th>
<th>(P) (N)</th>
<th>(n)</th>
<th>(M_o) (Nm)</th>
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5 Case Studies

Two mechanisms are employed as case studies to demonstrate the unique capabilities of the comprehensive solution to solve flexible beam problems that are outside the range of other methods.
5.1 Circular-Guided Compliant Mechanism. A circular-guided compliant mechanism shown in Fig. 18 has motion that is outside the range of other methods. Link OA is initially straight, flexible, and has one end attached to ground and the other rigidly connected to crank AB at point A. Crank AB is rigid and is pinned to ground at point B, thus guiding point A to follow a circular path. The lengths of link OA and AB are denoted as \( L_{OA} \) and \( L_{AB} \), respectively. The \( x \)-axis is oriented along link OA and the angle of link AB with respect to the \( x \)-axis is denoted as \( b \). The loop closure equations are given as

\[
\begin{align*}
a &= L_{OA} - L_{AB} \cos \beta_0 + L_{AB} \cos \beta \\
b &= -L_{AB} \sin \beta_0 + L_{AB} \sin \beta
\end{align*}
\]

where \( \beta_0 \) is the initial angle of link AB. The tip angle of beam OA is equal to \( \beta - \beta_0 \). The kinetostatics of the mechanism can be obtained by simultaneously solving Eqs. (20) and (30).

For the parameters of the mechanism given in Table 2, Fig. 19 shows the deflected shapes of link OA at different values of \( \beta \). The tip angle of beam OA is equal to \( \beta / C_0 \). The deflected curve of OA has two inflection points when \( \beta \) exceeds 164 deg, while the second one moves slightly around \( s = 2L/3 \).

Figures 20(b) and 20(c) show photographs of the deflected beam at \( \beta = 90 \) deg and \( \beta = 170 \) deg, respectively. They agree well with the results of the comprehensive solution shown in Fig. 19.

The kinetostatics of the mechanism are plotted in Fig. 21. Figure 21 shows that beam OA buckles at the beginning when \( nP \) reaches 47.7 N, which is slightly smaller than the critical buckling force predicted by Euler’s formula for long column with fixed ends \( (P_{cr} = 4\pi^2EI/L^2 = 48.23 N) \). This difference is due to the boundary conditions applied at the tip of OA.

5.2 Coupler-Curve Guided Compliant Mechanism. This subsection presents a case study of a coupler-curve guided compliant mechanism to show the capability of the comprehensive solution in solving large deflections of flexible beams with more than two inflection points. The mechanism is comprised of a four-bar crank-rocker mechanism and a cantilever beam (OE) whose free end is rigidly connected to the coupler of the crank-rocker mechanism via rigid link BE, as shown in Fig. 22.

The \( x \)-axis is oriented along link OE. As link AB rotates clockwise, the tip deflections and the tip angle of beam OE are given as

\[
\begin{align*}
a &= r_2(\cos \theta_2 - \cos \theta_20) + r_3(\cos(\theta_30 + \theta_1 - \theta_30) - \cos \theta_30) + L \\
b &= r_2(\sin \theta_2 - \sin \theta_20) + r_3(\sin(\theta_30 + \theta_1 - \theta_30) - \sin \theta_30) \\
\theta_0 &= \theta_1 - \theta_30
\end{align*}
\]

\[\]
where $\theta_{20}, \theta_{30},$ and $\theta_{50}$ are the initial angle of link AB, BC, and BE, respectively. The tip loads can be obtained by simultaneously solving Eqs. (20) and (31). Figure 23 shows the free-body diagram for link AB, BC, BE, and OE. Applying the static equilibrium for link AB, BC, and BE yields

\[
\begin{align*}
T_{in} &= F_{Bx} r_1 \sin \theta_1 - F_{By} r_2 \cos \theta_2 \\
F_{Bx} &= -n P - \frac{Pr_3 [\cos (\theta_{30} + \theta_3 - \theta_{30}) + n \sin (\theta_{30} + \theta_3 - \theta_{30})] + M_o}{r_3 (\tan \theta_4 \cos \theta_{30} - \sin \theta_3)} \\
F_{By} &= P - \frac{Pr_3 [\cos (\theta_{30} + \theta_3 - \theta_{30}) + n \sin (\theta_{30} + \theta_3 - \theta_{30})] + M_o}{r_3 (\tan \theta_4 \cos \theta_{30} - \sin \theta_3)} \tan \theta_4
\end{align*}
\]

For the parameters of the mechanism given in Table 3, point E traces a trajectory shown in Fig. 24. OE starts to be deflected with two inflection points. The third inflection point appears from the root of the beam when the moment at the root ($M_o$) changes sign from negative to positive. The results also show that the angles at the first and the third inflection points are equal when the beam contains three inflection points (e.g., inflection points A and C.

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**Table 3** Parameters of planar linkage mechanism

<table>
<thead>
<tr>
<th>$r_1$ (m)</th>
<th>$r_2$ (m)</th>
<th>$r_3$ (m)</th>
<th>$r_4$ (m)</th>
<th>$r_5$ (m)</th>
<th>$L$ (m)</th>
<th>$E$ (Pa)</th>
<th>$I$ (m$^4$)</th>
<th>$\theta_{20}$ (deg)</th>
<th>$\theta_{30}$ (deg)</th>
<th>$\theta_{40}$ (deg)</th>
<th>$\theta_{50}$ (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.052</td>
<td>0.13</td>
<td>0.117</td>
<td>0.115</td>
<td>0.111</td>
<td>1.4 x 10$^6$</td>
<td>4.9 x 10$^{-14}$</td>
<td>0</td>
<td>63.8</td>
<td>85.4</td>
<td>90</td>
</tr>
</tbody>
</table>
shown in Fig. 24). Figures 25(a) and 25(b) show photographs of the deflected beam with two inflection points and three inflection points, respectively. They agree well with the results of the comprehensive solution shown in Fig. 24. From the input torque curve plotted in Fig. 26, we can see that beam OE buckles at the beginning when \( n^2 \) reaches 0.2216N, which is approximately equal to the critical buckling force predicted by Euler’s formula for long column with fixed ends (\( P_{cr} = 4\pi^2EI/L^2 = 0.2242\text{N} \)).

6 Conclusions

A comprehensive solution based on the elliptic integrals was proposed for solving large deflection problems. By explicitly incorporating the number of inflection points (\( n \)) and the sign of the end-moment load (\( S_{ex} \)) in the derivation, the comprehensive solution is capable of solving large deflections of thin beams with multiple inflection points and subject to any kinds of end loads. The comprehensive solution also extends the elliptic integral solutions to be suitable for any beam end angle. A few deflected configurations of complex modes solved by the comprehensive solution were presented and discussed. Two examples taken from the literature were presented to verify the effectiveness of the comprehensive solution. Lastly, two mechanisms were employed as case studies to demonstrate the unique capabilities of the comprehensive solution to solve large deflection problems of flexible beams that are outside the range of other methods. The comprehensive solution also shows promise for use in compliant mechanism synthesis, which will be an important topic for our future work.

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