A new generalized model for elliptical arc flexure hinges

Guimin Chen,1,a) Xiaodong Shao,1 and Xinbo Huang2

1School of Mechatronics, Xidian University, Xi’an, Shaanxi 710071, People’s Republic of China
2School of Electronic Information, Xi’an Polytechnic University, Xi’an, Shaanxi 710071, People’s Republic of China

(Received 12 March 2008; accepted 10 August 2008; published online 8 September 2008)

Flexure hinges have been used in many engineering areas where high precision and sensitivity are required. Many kinds of flexure profiles were proposed during the past decade. Therefore, a general closed-form solution for flexure hinges of different profiles that incorporates the profile selection and parameter design will be of great benefit to the hinge design. The present work brings circular, right-circular, and elliptical profiles together by proposing a generalized flexure hinge model, which we call elliptical arc flexure hinges (whose maximum eccentric angle $\phi_m$ ranges from 0 to $\pi/2$) to distinguish from the existing elliptical flexure hinge ($\phi_m=\pi/2$). Based on the theories of mechanics of materials, all the elements in the compliance matrix for elliptical arc flexure hinges are deduced by introducing the eccentric angle of ellipse as the integral variable. These compliance equations simply boil down to four integrals, thus simplifying the compliance calculation. These equations also apply to elliptical ($\phi_m=\pi/2$), circular ($a=b$), and right-circular ($\phi_m=\pi/2$ and $a=b$) hinges.

These compliance equations were checked by comparing them with the results of finite element analysis, the existing equations, and experiment results. The comparison results show that these generalized equations are concise and adequate for most design purposes. © 2008 American Institute of Physics. [DOI: 10.1063/1.2976756]

I. INTRODUCTION

Flexure hinges in various forms find wide use in a variety of precision mechanisms such as micrograbbers, micropositioning stages, high-accuracy alignment devices, displacement amplifiers, and all kinds of parallel micro-mechanisms because they are small in size, high in sensitivity, and without mechanical friction and backlash.

Paros and Weisbord1 introduced circular flexure hinges [Fig. 1(c)] and provided both full theoretical and simplified calculation equations for them in 1965. A right-circular hinge [Fig. 1(a)] is a special circular hinge with the maximum central angle $\phi_m$ equal to $\pi/2$ (the cutouts are semicircular), therefore these equations also apply to right-circular hinges. In 2002, Wu and Zhou2 presented more concise equations for circular flexure hinges, which simplified the design calculation. Wu and Zhou’s equations have the same results as the exact ones provided by Paros and Weisbord, except the signs of $\alpha_x/F_y$ and $\Delta/M_z$ are opposite.

The advent of CNC wire electrodischarge machining technology made producing hinges of arbitrary cutout profiles easier, and consequently flexure hinges of various cutout profiles were introduced and studied. Lobontiu3 et al. presented the closed-form solutions for the in-plane compliance factors of corner-filleted flexure hinges, which were validated using finite element method and experiments. Lobontiu et al.4 introduced parabolic and hyperbolic flexure hinges and derived both in- and out-of-plane compliance formulas. Secant and inverse parabolic flexure hinges3 were introduced and studied as well.

There are several papers presenting design equations of elliptical flexure hinges [Fig. 1(b)]. Smith et al.5 derived closed-form compliance equations for elliptical flexure hinges by modifying the equations of Paros and Weisbord1 for right-circular flexure hinges. These equations were verified by both finite element analysis and experiment. Tseytlin7 presented thickness-dependent equations for elliptical hinges by following the method of inverse conformal mapping. A more tractable but less accurate compliance equation for elliptical flexures is provided in this work as well. Lobontiu4 derived the in-plane compliance equations for elliptical hinges based on Castigliano’s second theorem. His equations have the same results as the ones provided by Smith et al. However, the elliptical flexure hinges studied in these papers are formed by cutting two semieliptical cutouts symmetrically, namely, the maximum eccentric angle of the cutouts $\phi_m=\pi/2$, as shown in Fig. 1(b). Flexure hinges with elliptical arc cutouts of arbitrary maximum eccentric angle $\phi_m$ have not been studied so far. This paper expands upon the previous work by presenting flexure hinges with elliptical arc cutouts of arbitrary $\phi_m$ ($\phi_m$ ranges from 0 to $\pi/2$), which we call elliptical arc flexure hinges to distinguish from the existing elliptical flexure hinge. An elliptical arc flexure hinge is shown in [Fig. 1(d)].

In the vast family of flexure hinges, right-circular, circular, and elliptical flexures are most commonly used in flexure-based precision instruments. For a designer, it is burdensome to work with all these types of flexure hinges and their corresponding design equations in order to optimize instrument design. Experiments suggest that for hinge profiles of the same cutout length and same minimum width,
right-circular hinges provide the most precise motion, elliptical hinges offer better results with regard to stress conditions (thus long fatigue life), and circular hinges provide a combination of these characteristics. Because this knowledge always drives the selection of different hinge profiles in instrument design, designers often choose hinge profiles without knowing for certain which one works best.6-10 This situation impedes designers from finding the most suitable hinge designs for instruments. The elliptical arc flexure hinges presented in this paper provide a solution to this problem in the form of general closed-form equations, which encompass and expand upon circular, right-circular, and elliptical flexure hinges. This greatly simplifies the process of cutout profile selection and parameter design for the designer, making it possible to consistently find and utilize optimal hinge designs.

In a nutshell, we extend the definition of elliptical hinges by changing the parameter $\phi_m$ of the cutout profiles from $\phi_m = \pi/2$ (semiellipse) to $0 < \phi_m \leq \pi/2$ and correspondingly call them elliptical arc flexure hinges. By doing so, we derive a generalized compliance model for circular, right-circular, and elliptical flexure hinges, bringing all these hinge types together under one set of equations. It should also be noted that by extending $\phi_m$ to $0 < \phi_m \leq \pi/2$, this model extends the design domain of the profile beyond the scope of the other three kinds of flexure hinges.

In Fig. 2, the relationship between circular, right-circular, elliptical, and elliptical arc flexure hinges is shown, where the former three are subsets of the latter, with right-circular hinges ($a=b$ and $\phi_m=\pi/2$) forming the intersection between circular ($a=b$) and elliptical hinges (the eccentric angle becomes the central angle as $a=b$). All the closed-form compliance equations for elliptical arc flexure hinges are derived by introducing the eccentric angle as the integral variable, which makes them more concise and effective. According to Fig. 2, these equations must also apply to elliptical flexure hinges when $a=b$, circular flexure hinges when $a=b$, and right-circular flexure hinges when $a=b$ and $\phi_m=\pi/2$, allowing designers to utilize optimizing technique for hinge configuration design conveniently. Figure 3 shows a three-dimensional diagram of an elliptical arc hinge.

II. ECCENTRIC ANGLE OF AN ELLIPSE

Figure 4 shows an ellipse and its circumscribed circle of radius $a$ and inscribed circle of radius $b$. We can obtain the eccentric angle corresponding to any point $P$ on the ellipse by drawing a vertical line through $P$, which intersects the circumscribed circle at $L$ above $P$ (or below if $P$ is on the lower semiellipse). This is $\varphi$, the angle between the radius at point $L$ and the positive $x$ axis (one of the axes of the ellipse). Moreover, the radius at $L$ intersects the inscribed circle at $N$. Because the ellipse can be represented in a parametric form as
\[ x = a \cos \varphi, \quad y = b \sin \varphi (0 \leq \varphi < 2\pi), \]  
we know that point \( P \) has the same abscissa as point \( L \) on the circumscribed circle and the same ordinate as point \( N \). Therefore, for any point \( P \) on the ellipse, its coordinates are given by
\[ x_P = a \cos \varphi, \quad y_P = b \sin \varphi (0 \leq \varphi < 2\pi). \]  
The cutout profile of an elliptical arc hinge can be determined by the cutout length \( a \) (as shown in Fig. 3), the ellipse semiaxis along the cutout depth \( b \), and the maximum eccentric angle \( \phi_m \). The ellipse semiaxis along the cutout length \( a \) is given by
\[ a = \frac{c}{\sin \phi_m}. \]  
For the elliptical arc hinge shown in Fig. 5, as the eccentric angle \( \phi \) traverses from \( -\phi_m \) to \( \phi_m \), the corresponding point on the elliptical arc cutout varies from \( P_1 \) to \( P_2 \). Besides, when \( a = b \), the ellipse becomes a circle, and the eccentric angle becomes a central angle.

### III. Assumptions

The derivation of the compliance equations is based on the following assumptions.

- For most applications, the minimum thickness of the flexure hinges \( t \) is much smaller than the cutout length \( l \), so the hinge can be treated as a beam and the Euler–Bernoulli beam theory is applicable; that is, the approximate compliances of hinges can be obtained by integrating the linear differential equations of a beam based on mechanics of materials.
- It is assumed that the deformations of flexure hinges in all the directions are so small that superposition applies when a combination of loads is applied.
- Assume a flexure hinge as a six degree-of-freedom fixed-free beam subjected to bending, axial loading, shearing, and torsion, as shown in Fig. 3. By defining the force acting on the hinge by
\[ \mathbf{F} = [F_x, F_y, F_z, M_x, M_y, M_z]^T \]  
and the corresponding deformations of the hinge by
\[ \mathbf{X} = [\Delta_x, \Delta_y, \Delta_z, \alpha_x, \alpha_y, \alpha_z]^T, \]  
the following relationship is obtained:  
\[ \mathbf{X} = \mathbf{C}_h \mathbf{F}. \]  
where \( \mathbf{C}_h \) is the compliance matrix of the hinge, which can be expressed by
\[
\mathbf{C}_h = \begin{bmatrix}
\frac{\Delta_x}{F_x} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{\Delta_y}{F_y} & 0 & 0 & \frac{\Delta_z}{M_z} & 0 \\
0 & 0 & \frac{\Delta_z}{M_z} & 0 & \frac{\Delta_x}{F_x} & 0 \\
0 & 0 & 0 & \frac{\alpha_x}{M_x} & 0 & 0 \\
0 & 0 & \frac{\alpha_y}{M_y} & 0 & \frac{\alpha_z}{M_z} & 0 \\
0 & \frac{\alpha_z}{M_z} & 0 & 0 & \frac{\alpha_y}{M_y} & \frac{\alpha_x}{M_x}
\end{bmatrix}.
\]  
Each element in the compliance matrix is derived from beam theory in the following. We also take the directions of forces, moments, and deflections into consideration. Once the compliance matrix is obtained, the stiffness matrix of the hinge can be derived by
\[ \mathbf{K}_h = \mathbf{C}_h^{-1}. \]  

### IV. Compliance Equations for Elliptical Arc Flexure Hinges

Slice the profile of the hinge region into thin vertical infinitesimal strips. The height of the infinitesimal strip \( dx \) at position \( x \) for an elliptical arc hinge can be expressed as (as shown in Fig. 5)
\[ h(x) = 2b + t - 2\frac{b}{a}\sqrt{a^2 - (c-x)^2}. \]  
By introducing the eccentric angle, we have
\[ x = c + a \sin \phi, \]  
and Eq. (9) can be rearranged to yield
\[ h(\phi) = 2b + t - 2b \cos \phi. \]  
Differentiation of Eq. (10) yields
\[ dx = d(a \sin \phi) = a \cos \phi d\phi. \]  
Supposing \( s = b/t \) and substituting it into Eq. (11), the strip height at position \( \phi \) can be expressed as
\[ g(\phi) = h(\phi)/b = 2 + 1/s - 2 \cos \phi. \]  

#### A. Angular compliance about the z axis

The \( z \) axis is the axis that outputs the desired rotation (also called flexible axis), so the angular compliance about the \( z \) axis is the most important parameter of the design of flexure hinge.

We denote the angular deflection about the \( z \) axis by \( \alpha_z \) and \( \alpha_z \) may be caused by moment \( M_z \) or force \( F_z \). According to the Euler–Bernoulli beam theory, \( \alpha_z \) caused by moment \( M_z \) can be expressed as
where $E$ is the elastic modulus of the material and $I_x(x)$ is the cross-sectional area moment of inertia about the neutral axis at the given position $x$. By substituting $x$ with eccentric angle $\phi$, the corresponding compliance can be expressed as

$$\frac{\alpha_x}{M_z} = \int_{-\phi_m}^{\phi_m} \frac{a \cos \phi}{E[w^3(h(\phi))/12]} d\phi = \frac{12a}{Ew^3b} \int_{-\phi_m}^{\phi_m} g^3(\phi) d\phi$$

$$= \frac{12a}{Ew^3b} N_2. \quad (15)$$

As to the force $F_y$, its contribution to $\alpha_x$ equals its equivalent moment $(c-a \sin \phi)F_y$ about the $z$ axis. Therefore, $\alpha_x$ caused by force $F_y$ is given by

$$\alpha_x = \int_{-\phi_m}^{\phi_m} (c-a \sin \phi)(a \cos \phi)F_y \frac{1}{E[w^3(h(\phi))/12]} d\phi$$

$$= \int_{-\phi_m}^{\phi_m} \frac{acF_y \cos \phi}{E[w^3(h(\phi))/12]} - \frac{a^2F_y \sin \phi \cos \phi}{E[w^3(h(\phi))/12]} d\phi. \quad (16)$$

The second term in the above integrand is an odd function, so its integral has the value of 0. Therefore, the corresponding compliance is

$$\frac{\alpha_x}{F_y} = \frac{12ac}{Ew^3b} \int_{-\phi_m}^{\phi_m} \frac{\cos \phi}{g^3(\phi)} d\phi = \frac{12ac}{Ew^3b} \int_{-\phi_m}^{\phi_m} \frac{\cos \phi}{g^3(\phi)} d\phi = \frac{12ac}{Ew^3b} N_2. \quad (17)$$

**B. Angular compliance about the $y$ axis**

Both moment $M_y$ and force $F_y$ may cause the angular deflection $\alpha_y$ about the $y$ axis. According to the beam theory, $\alpha_y$ caused by moment $M_y$ can be expressed as

$$\alpha_y = \int_{0}^{2\pi} \frac{M_y}{EI_y(x)} dx = \int_{0}^{2\pi} \frac{M_y}{E[h(x)w^3/12]} dx. \quad (18)$$

By substituting $x$ with the eccentric angle $\phi$, the corresponding compliance can be expressed as

$$\frac{\alpha_y}{M_y} = \int_{-\phi_m}^{\phi_m} \frac{a \cos \phi}{E[w^3(h(\phi))/12]} d\phi = \frac{12a}{Ew^3b} \int_{-\phi_m}^{\phi_m} g^3(\phi) d\phi$$

$$= \frac{12a}{Ew^3b} N_1. \quad (19)$$

As to force $F_z$, its contribution to $\alpha_y$ equals its equivalent moment $-(c-a \sin \phi)F_z$ about the $y$ axis. Therefore, $\alpha_y$ caused by force $F_z$ is given by

$$\alpha_y = \int_{-\phi_m}^{\phi_m} \frac{(c-a \sin \phi)(a \cos \phi)F_z}{E[w^3(h(\phi))/12]} d\phi$$

$$= \int_{-\phi_m}^{\phi_m} \frac{a^2F_z \sin \phi \cos \phi}{E[w^3(h(\phi))/12]} - \frac{acF_z \cos \phi}{E[w^3(h(\phi))/12]} d\phi. \quad (20)$$

Again, the first term in the above integrand is an odd function, so the corresponding compliance can be written as

$$\frac{\alpha_y}{F_z} = \frac{12ac}{Ew^3} \int_{-\phi_m}^{\phi_m} \frac{\cos \phi}{g^3(\phi)} d\phi = \frac{12ac}{Ew^3b} N_1. \quad (21)$$

**C. Linear compliance along the $z$ axis**

Linear deflection $\Delta$ may be caused by moment $M_z$, force $F_z$, due to moment $M_y$ can be expressed as

$$\Delta_z = \int_{0}^{2\pi} \frac{dz}{dx} dx = -\int_{0}^{2\pi} \alpha_y M(x) dx$$

$$= -\int_{-\phi_m}^{\phi_m} \alpha_y^M(\phi) d(a \sin \phi), \quad (22)$$

For

$$dx^M_y(\phi) = \frac{M_y a \cos \beta}{E[w^3h(\beta)/12]} d\beta, \quad (23)$$

we have

$$\Delta_z = \left[ a \sin \phi \int_{-\phi_m}^{\phi_m} \frac{M_y a \cos \beta}{E[w^3h(\beta)/12]} d\beta \right]_{-\phi_m}^{\phi_m}$$

$$+ \int_{-\phi_m}^{\phi_m} \frac{M_y a^2 \cos \phi \sin \phi}{E[w^3h(\beta)/12]} d\phi. \quad (25)$$

The integrand in the second term of Eq. (25) is an odd function, thus the compliance is

$$\Delta_z = \frac{12ac}{Ew^3b} \int_{-\phi_m}^{\phi_m} \frac{\cos \beta}{g(\beta)} d\beta = \frac{12ac}{Ew^3b} N_1. \quad (26)$$

$\Delta$ caused by force $F_z$ can be divided into two parts, one part is due to bending, the other is due to shearing. The bending part is

$$\Delta^b_y = \int_{-\phi_m}^{\phi_m} \alpha_y^b(\phi) d(a \sin \phi), \quad (27)$$

where

$$\alpha_y^b(\phi) = \int_{-\phi_m}^{\phi_m} \frac{(c-a \sin \phi)F_z a \cos \beta}{E[w^3h(\beta)/12]} d\beta. \quad (28)$$

For

$$d\alpha_y^f(\phi) = \frac{12ac}{Ew^3b} N_1, \quad (29)$$

then
\[ \Delta_y = \int_{-\phi_m}^{\phi_m} a \sin \phi \frac{M_\alpha \cos \beta}{Ewh^3(\beta)/12} d\beta \]

and the shearing part is
\[ \Delta_s = \int_{-\phi_m}^{\phi_m} \alpha_s^x(\phi) d(a \sin \phi), \]

where \( G \) and \( k \) are the shearing modulus and shearing coefficient of the material, respectively. For a beam of rectangular cross-section, \( k \) is given by
\[ k = \frac{12 + 11 \nu}{10 + 10 \nu}, \]

where \( \nu \) is Poisson's ratio, and
\[ \nu = \frac{E}{2G} - 1. \]

To sum up, the linear compliance due to \( F_z \) is
\[ \frac{\Delta_y}{F_z} = \frac{\Delta_s}{F_z} = \left( \frac{12ac^2 + 12a^3}{Ew^3b} + \frac{ka}{Gwb} \right) \frac{p}{N} = \frac{12a^3}{Ew^3b} \frac{N_3}{N_1}. \]

D. Linear compliance along the \( y \) axis

Linear deflection \( \Delta_y \) along the \( y \) axis may be caused by moment \( M_y \) or force \( F_y \). \( \Delta_y \) due to moment \( M_y \) can be expressed as
\[ \Delta_y = \int_{-\phi_m}^{\phi_m} \alpha_y^M(\phi) d(a \sin \phi), \]

where
\[ \alpha_y^M(\phi) = \int_{-\phi_m}^{\phi} \frac{M_\alpha \cos \beta}{E[wh^3(\beta)/12]} d\beta. \]

For
\[ d\alpha_y^M(\phi) = \frac{M_\alpha \cos \beta}{E[wh^3(\beta)/12]} d\beta, \]

then
\[ \Delta_y = \int_{-\phi_m}^{\phi_m} a \sin \phi \frac{M_\alpha \cos \beta}{Ewh^3(\beta)/12} d\beta \]

\[ \int_{-\phi_m}^{\phi_m} M_\alpha \cos \beta \frac{\phi_m}{Ewh^3(\beta)/12} d\beta = \frac{12ac}{Ew^3b} \int_{-\phi_m}^{\phi_m} \frac{\phi_m}{wh^3(\beta)/12} d\beta = \frac{12ac}{Ew^3b} N_2. \]

The integrand in the second term of Eq. (38) is an odd function, thus the compliance is
\[ \frac{\Delta_y}{M_y} = \frac{a \sin \phi}{Ewh^3(\beta)/12} d\beta = \frac{12ac}{Ew^3b} \int_{-\phi_m}^{\phi_m} \frac{\phi_m}{wh^3(\beta)/12} d\beta. \]

\( \Delta_s \), caused by force \( F_s \), can be divided into two parts, one part is due to bending, the other is due to shearing. The bending part is
\[ \Delta_b = \int_{-\phi_m}^{\phi_m} \alpha_s(\phi) d(a \sin \phi), \]

where
\[ \alpha_s(\phi) = \frac{c - a \sin \beta}{Ewh^3(\beta)/12} d\beta. \]

Differentiation of \( \alpha_s(\phi) \) yields
\[ d\alpha_s(\phi) = \frac{aF_s(c - a \sin \phi) \cos \psi}{E[wh^3(\beta)/12]} d\phi, \]

thus,
\[ \Delta_s = \int_{-\phi_m}^{\phi_m} a \sin \phi \frac{M_\alpha \cos \beta}{Ewh^3(\beta)/12} d\beta \]

\[ \int_{-\phi_m}^{\phi_m} a \sin \phi \frac{M_\alpha \cos \beta}{Ewh^3(\beta)/12} d\beta = \frac{12ac}{Ew^3b} \int_{-\phi_m}^{\phi_m} \frac{\phi_m}{wh^3(\beta)/12} d\beta. \]

And the shearing part is
\[ \Delta_s = \frac{kaw}{Gwb} N_1. \]

To sum up, the linear compliance due to \( F_y \) is
\[ \frac{\Delta_y}{F_y} = \frac{(\Delta_b + \Delta_s)}{F_y} = \frac{12ac^2 + 12a^3}{Ew^3b} \frac{N_3}{N_2} = \frac{12a^3}{Ew^3b} N_4 \frac{ka}{Gwb} N_1. \]

E. Linear compliance along the \( x \) axis

We denote the linear deflection caused by tension load \( F_z \) by \( \Delta_x \), which can be expressed as
\[ \Delta_c = \int_0^{2\pi} \frac{F_s}{Ewh(x)} dx = \int_{-\phi_m}^{\phi_m} \frac{F_s a \cos \phi}{Ewh(\phi)} d\phi, \]  
so the linear compliance along the \( x \) axis is

\[ \frac{\Delta_c}{F_c} = \frac{a}{Ewh} \int_{-\phi_m}^{\phi_m} \cos \phi g(\phi) d\phi = \frac{a}{Ewh} N_1. \]

\[ (46) \]

\[ (47) \]

\[ \text{F. Angular compliance about the } x \text{ axis} \]

Denote the torsion angle due to moment \( M_x \) by \( \alpha_x \). Because each infinitesimal strip of the hinge can be treated as a constant rectangular cross-section beam, according to the approximation equation given in Hearn’s book,\(^{13}\) \( \alpha_x \) can be expressed as

\[ \alpha_x = \int_0^{2\pi} \frac{42M_x I_t(x)}{Gw^3 h^3(x)} dx, \]

\[ (48) \]

where \( I_t(x) \) is the torsional inertia moment of the infinitesimal strip at position \( x \) and \( I_t(x) = [w^4 h(x) + wh^3(x)]/12 \). By substituting Eqs. (12) and (13) into Eq. (48), the torsional compliance can be expressed as

\[ \frac{\alpha_x}{M_x} = \frac{7a}{2Gw^3 b} \int_{-\phi_m}^{\phi_m} \cos \phi g(\phi) d\phi \]

\[ = \frac{7a}{2Gw^3 b} N_1 + \frac{7a}{2Gw^3 b} N_2. \]

(49)

It should be noted that \( \frac{\alpha_c}{F_c} = \Delta_c/M_c \) and \( \frac{\alpha_c}{F_c} = \Delta_c/M_c \), therefore the compliance matrix is a symmetric matrix, that is to say, there are eight independent design parameters in the matrix.

\[ N_1 = \int_{-\phi_m}^{\phi_m} \cos \phi g(\phi) d\phi \]

\[ = 2(2s + 1) \arctan \left( \frac{\phi_m}{2} \right) - \phi_m, \]

(50)

\[ N_2 = \int_{-\phi_m}^{\phi_m} \cos \phi g(\phi) d\phi \]

\[ = 12s^4 (2s + 1) \arctan \left( \frac{\phi_m}{2} \right) + 2s^3 (2s + 1) (6s^2 + 4s + 1) \sin \phi_m \]

\[ + \frac{(4s + 1)^2 (1 + 2s - 2s \cos \phi_m)^2}{2s} - 2s^3 (12s^2 + 4s + 1) \sin \phi_m \cos \phi_m \]

\[ - \frac{(4s + 1)^2 (1 + 2s - 2s \cos \phi_m)^2}{2s} \]

(51)

\[ N_3 = \int_{-\phi_m}^{\phi_m} \cos^3 \phi g(\phi) d\phi \]

\[ = \frac{(2s + 1)^3}{2s^2} \arctan \left( \frac{\phi_m}{2} \right) - \frac{2s + 1 + s \cos \phi_m}{2s} \sin \phi_m - \frac{6s^2 + 4s + 1}{4s^2} \phi_m, \]

(52)

\[ (53) \]

\[ \text{VI. VERIFICATION OF THE COMPLIANCE EQUATIONS} \]

The finite element software ANSYS was used to check the generalized compliance equations for elliptical arc flexure hinges under static loading. The physical and geometric parameters of five elliptical arc flexure hinges, including an elliptical one, a circular one, and a right-circular one, are listed in Table I. The finite element models (FEMs) of the flexure hinges were generated using four-node, three-

\[ \text{TABLE I. Several design examples of flexure hinges. For all the designs } E=2.07 \times 10^{11} \text{ N/m}^2, \text{ } G=8.1 \times 10^{10} \text{ N/m}^2, \text{ } w=10 \text{ mm, } t=1 \text{ mm, and } c=5 \text{ mm.} \]

<table>
<thead>
<tr>
<th>Example</th>
<th>Type</th>
<th>( \phi_m ) (deg)</th>
<th>( a ) (mm)</th>
<th>( b ) (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Elliptical Arc</td>
<td>45</td>
<td>7.071</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>Elliptical Arc</td>
<td>60</td>
<td>5.774</td>
<td>4</td>
</tr>
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<td>4</td>
</tr>
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<td>Circular</td>
<td>60</td>
<td>5.774</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>Right-circular</td>
<td>90</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

![FIG. 6. (Color online) FEM.](image-url)
dimensional, solid elements (Solid72) with six degree-of-freedom per node. The smart meshing method, which automatically refines meshes in higher stress concentration regions, was used. Each hinge model is fixed on one end, and loaded with forces \( F_x, F_y \), and \( F_z \) and moments \( M_x, M_y \), and \( M_z \) on the opposite end, respectively. One of the hinge models is shown in Fig. 6. Table II comprises the analytical and finite element results. As shown in Table II, the analytical and finite element results are in good agreement. The relative errors between the analytical and finite element results are less than 8%.

Table II also presents the results of both the exact equations of Smith \textit{et al.} for the elliptical hinge and the exact equations of Paros and Weisbord for the circular and right-circular hinges (Smith \textit{et al.} derived four compliance equations for elliptical flexure hinges, namely, \( \Delta_x/F_x \), \( \Delta_y/F_y \), \( \alpha_{x}/M_x \), and \( \alpha_{y}/M_y \), and Paros and Weisbord did not give the equation for torsional compliance \( \alpha_{z}/M_z \)). These results are the same as the ones using the generalized equations of this paper, except \( \Delta_x/F_x \) and \( \Delta_y/F_y \), because we take the shearing coefficient (k) into consideration in Eq. (44). Compared to the finite element results, the results of the generalized equations for \( \Delta_x/F_x \) and \( \Delta_y/F_y \) are more accurate than those of the equations of Smith \textit{et al.} and Paros and Weisbord.

Experimentation was also used to assess the validity of the angular compliance equation about the input axis \( \alpha_{z}/M_z \). Five elliptical arc flexure hinges made of 45 grade steel were machined by using wire electrodischarge machining technology according to the geometric parameters listed in Table I. Each hinge was machined as an “I” shaped sample with the upper and lower horizontal bars being separated by the hinge itself and a 45° wedge on the upper bar exactly.

![FIG. 7. (Color online) The experimental setup.](Image)
above the hinge. The experimental setup comprises an optical platform, a 6500 A laser diode, a reflector (mounted on the wedge), a position sensitive detector (PSD), and the hinge sample, as shown in Fig. 7. The lower bar is fixed to the optical platform, and the diode laser mounted immediately above the reflector that reflects a laser beam to the PSD. To improve the measurement precision, the distance between the PSD and the test hinge should be as long as possible. In the experiments, we put the PSD 500 mm away from the hinges. For each hinge, we loaded a mass of 200 g on both sides of the upper bar to simulate two bending load cases, so two compliance values could be obtained. Therefore, the corresponding angular compliance was calculated as an average. It should be noted that this loading case will give an additional compression force to the sample; however, the potential error sources associated with this configuration, such as hinge compression and parasitic deflections, are negligible, as discussed by Smith et al. When the hinge sample deformed due to the load, the PSD read the displacement of the laser spot on itself, which was used to calculate the angular deflection of the sample. The experimental results are summarized in Table III. The results are within 7% error compared to the analytical results in Table II.

It should be noted that although an elliptical arc flexure hinge is designed to primarily have large compliance about its input axis ($\alpha_z/M_z$), the torsional compliance ($\alpha_x/M_x$) is always comparable in magnitude. Therefore, the torsional compliance of elliptical arc flexure hinges should be treated carefully and used adequately.

**ACKNOWLEDGMENTS**

The authors would like to thank Daniel B. Seegmiller for his help with technical writing, as well as the reviewers and the editors for their valuable comments. This research is supported by the National Natural Science Foundation of China under Grant No. 50805110 and the China Postdoctoral Science Foundation under Grant No. 20070421110.

**TABLE III.** Experimental measurements of elliptical arc hinge compliances.

<table>
<thead>
<tr>
<th>$\alpha_z/M_z$</th>
<th>% error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 $2.437 \times 10^{-2}$</td>
<td>$-5.02$</td>
</tr>
<tr>
<td>2 $1.861 \times 10^{-2}$</td>
<td>$2.26$</td>
</tr>
<tr>
<td>3 $1.538 \times 10^{-2}$</td>
<td>$6.9$</td>
</tr>
<tr>
<td>4 $1.529 \times 10^{-2}$</td>
<td>$4.84$</td>
</tr>
<tr>
<td>5 $1.443 \times 10^{-2}$</td>
<td>$3.12$</td>
</tr>
</tbody>
</table>