Distributed Concatenated Recursive Alamouti-Circulant STBC for Two-Way Multi-Relay Networks

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***Abstract*— A general distributed multi-antenna two-way relaying network with multiple relays is considered in this paper. We first obtain a tight lower bound of pairwise error probability (PEP) of a maximum likelihood detector for the general distributed linear dispersion code for a half-duplex amplify-and-forward two-way relaying network (TWRN) consisting of two sources with each having single antenna and N relays with each having two antennas. Furthermore, by jointly considering signal precoding at the sources and signal processing at the relays, a general distributed concatenated recursive Alamouti-circulant space-time block code is proposed for the considered TWRNs. Our design ensures that the equivalent channel matrices at both source nodes are the so-called the recursive Alamouti-circulant matrices, with each block being a product of the two Alamouti channel matrices. Based on a lower–upper bound strategy and an induction method, asymptotic PEP formula is attained to show that given the optimal angle rotation matrix and the precoding matrix, the code can meet the lower bound of the diversity gain, as well as the maximum coding gain. In addition, the proposed rate one code turns to be effectively decodable.**

***Index Terms*— Two-way relaying networks (TWRNs), distributed space-time block code (DSTBC), pairwise error probability (PEP), recursive Alamouti-Circulant (RAC) matrices, diversity gain.**

I. Introduction

WIRELESS relaying communication is widely recognized as a promising candidate for more reliable broad area transmission in future communication systems [1], [2]. Compared with the traditional one-way relaying with half-duplex relays, the two-way relaying with higher spectral efficiency is proposed in [3], which is accomplished by simultaneously transmitting from sources to the relays in the multi-access phases and broadcasting the processed information from relays to sources in the broadcast phases. Two-way relaying networks (TWRNs) has been widely studied in remote communication, mobile communication and satellite communication [1], [2], [4], [5], etc. One of the principle tasks in TWRNs is to enhance the transmission reliability between the sources and the relays. Using multiple antennas or distributed multiple antennas come to effective solutions to this task. With a competitive diversity performance, the space-time block code (STBC) was proposed for the multi-input multi-output (MIMO) systems, such as Alamouti STBC, orthogonal and quasi-orthogonal STBC, block-orthogonal STBC, group-decodable STBC [6]–[9], etc. Furthermore STBC has also been wisely considered in the TWRNs [10], [11], i.e, distributed STBCs (DSTBCs). DSTBCs have attracted great attentions since they can reduce the complicated realization at the transmitting node and the receiving node while achieving full diversity. Specially, in view of the promising spectrum efficiency gain of TWRNs, many DSTBC-based TWRN protocols have also been reported [12]–[28]. These significant studies have strongly characterize DSTBC-based TWRNs as follows:

- Being great different with the MIMO system, the channel in the relay system is generally non-linear, and the statistics of the equivalent received noises at sources strongly depend on the channels as well as on the signal processing at the relay nodes.
- Distributed space-time coding for the relay system is jointly performed through the source nodes and relay nodes with corrupted noisy signals. Specially, the signals from different relay nodes cannot be exchanged.
- From the view of theoretical analysis, the diversity gain for the relay system involves the logarithm of SNR [11], [24], [25], [29] if the amplify-and-forward (AF) protocols are considered.
- Given the total network power, power loading among the source nodes and the relay nodes significantly affects the overall performance of the relay system [11], [25], [30]–[32].

The results regarding to the DSTBC relaying networks in the state of the art mostly come from the theoretical performance analysis and the feasible DSTBC design for some specific scenarios. Generally speaking, because of the intractability of the theoretical performance derivation in the distribution relaying networks, most accurate performance evaluation in mathematically cannot be attained in technically straightforward manner by just following the MIMO techniques. To the best knowledge of the authors, only the upper

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bound of the diversity gain for a general relaying network was derived [11], [29], whereas asymptotic performance analysis is available only for some specific relaying protocols [12]–[14], [24], [25], [28], [33]–[39]. However, the upper bound cannot tell us what is either the best diversity gain or the optimal coding gain for a DSTBC be possibly provided. From the view of the implementation, i.e. the concrete DSTBC design, it is still not clear that which DSTBC design is optimal in many cases. Therefore, there exist several aspects to be explored even for conventional STBC. For example, although many STBCs with rate-one have been proposed for as many as eight transmit antennas [40]–[43], however, computer searching will be required for those codes to obtain the optimal parameters for the specific modulation constellations. In addition, to fully explore the advantages of multi-antenna deployment and to enhance the link reliability, two or more nodes may be required to work as relay nodes. However, as far as we know, only the specific system models with one or two relay nodes are considered in most of the current reported studies and they cannot be extended to more general cases straightforwardly.

In this paper, we focus on the half-duplex TWRNs as those considered in [24], [25], and [30], in which each source has single antenna and every relay is equipped with two antennas. By utilizing the received signals from the two antennas located on the same relay node, we investigate how to design a novel DSTBC to improve the diversity performance, whilst keeping the low-complexity property of orthogonal STBC and quasi-orthogonal STBC. We mainly consider the more general TWRNs with multiple dual-antenna relays and its optimal DSTBC design. The main contributions of our work are summarized as follows:

1) For the first time, a tight lower bound on PEP with the ML detector for any distributed linear dispersion code with a fixed-gain amplifier is established in Section II for the general TWRN with N dual-antenna relays, showing that the diversity gain cannot decay faster than $\ln^N \mathrm{SNR}/\mathrm{SNR}^{2N}$, whereas for the similar network with $2N$ single-antenna relays, error performance decays with $\ln^{2N} \mathrm{SNR}/\mathrm{SNR}^{2N}$ [11], [29].
2) A general design with its decoding delay being half or less than that of the design in [30] is proposed in Section III. The transmission block having $2N$ complex signals is firstly divided into two groups with each including N symbols according to the parity of the indices. Then, both groups are rotated and precoded individually before they are combined into the transmit signal. We also give the optimal angle rotation matrix and precoding matrix to ensure the full rank of the difference matrix and maximize its determinant when M-QAM constellations are used.
3) The notable results in [24] and [25] are extended to a more general case, i.e., the TWRNs with 2^b relays. Instead of obtaining the closed-form PEP by using the direct analysis strategies [24], [25], we investigate the asymptotic PEP performance in Section IV by deriving the upper bound and the lower bound, as well as utilizing the induction method. Asymptotic PEP analysis shows that our design meets the diversity-gain lower bound as well as achieves the maximum coding gain for quadrature amplitude modulation (QAM).
4) The optimal power allocation that maximizes the received SNR of the worse link is attained in Section III and examined by measuring average bit error rate (BER) performance of the whole system through comprehensive computer simulations for asymmetric relaying channels.

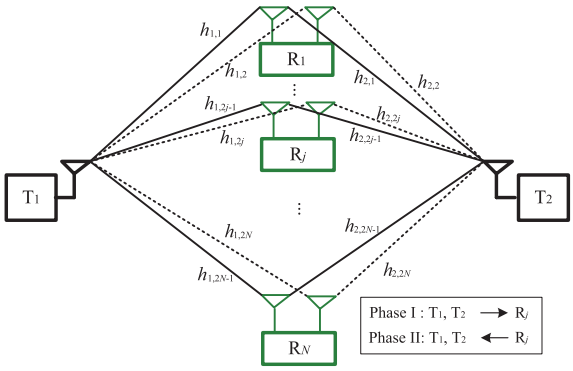

Fig. 1. The TWRN model with N dual-antenna relays.

Notation: Notation $\|\mathbf{r}\|$ denotes a 2-norm of the vector $\mathbf{r}$, $\mathcal{R}(\mathbf{r}) = [\Re(\mathbf{r}^T), \Im(\mathbf{r}^T)]^T$, where $\Re$ and $\Im$ represent the real part and imaginary part of a complex vector, respectively; $\mathbf{A}^T$, $\mathbf{A}^*$, $\mathbf{A}^H$, $\det(\mathbf{A})$, $\mathrm{vec}(\mathbf{A})$ denote transpose, conjugate, conjugate transpose, determinant, and vectorization of the matrix $\mathbf{A}$ respectively; $\mathbb{E}[\cdot]$ denotes the expected value of the expression in brackets; $\mathbf{I}_N$ denotes the $N \times N$ identity matrix. Hadamard(N) is a Hadamard matrix of order N; $\mathbf{A} \succeq \mathbf{B}$ denotes that $\mathbf{A}$ and $\mathbf{B}$ are positive semi-definite and $\mathbf{A} - \mathbf{B}$ is also positive semi-definite; We use $\otimes$ and $\odot$ denote the Kronecker product and the Schur product; Notation $f(x) = O(g(x))$ with $g(x) \geq 0$ denotes that there exists a pair of constants, c_1 and c_2, independent of the variable x such that $c_1 g(x) \leq f(x) \leq c_2 g(x)$; $(n)!!$ is the double factorial notation; We also use $\mathrm{diag}(a_1, a_2, \cdots, a_n)$ to represent a diagonal matrix whose diagonal entries are $a_1, a_2, \cdots, a_n$.

II. Two-Way Communication With Multiple Dual-Antenna Relays

In this paper, we consider a half-duplex AF TWRN consisting of two sources with each having single antenna and N relays with each having two antennas. As shown in Fig. 1, the two-way relaying transmission can be described as follows.

In the $2N$ time slots of the first phase, T_k, $k = 1, 2$ transmit their messages $\mathbf{s}_k = \left[s_{k,1}, s_{k,2}, \cdots, s_{k,2N}\right]^T$ to all relay nodes, with the transmission power being P_k. The received signal vector $\mathbf{r}_{j,l} = [r_{j,l}^{(1)}, r_{j,l}^{(2)}]^T$ from the two antennas of R_j at the l-th time slot can be written as

$$\mathbf{r}_{j,l} = \sqrt{P_1}\mathbf{h}_{1,j} s_{1,l} + \sqrt{P_2}\mathbf{h}_{2,j} s_{2,l} + \mathbf{n}_{j,l}, \tag{1}$$

where $\mathbb{E}[\mathbf{s}_k \mathbf{s}_k^H] = \mathbf{I}_{2N}$, $l = 1, 2, \cdots, 2N$. Furthermore, $\mathbf{h}_{k,j} = [h_{k,2j-1}, h_{k,2j}]^T$ denotes the the channel vector between T_k and R_j, $j = 1, \cdots, N$. We assume $h_{k,l}$ are independent non-identically distributed (*i.n.i.d*) complex Gaussian random variables with zeros-mean and variances $\mathbf{\Omega}_k$.

We further assume $h_{k,l}$ remain constant in one transmission block. In addition, we also assume that perfect channel state information (CSI) $h_{k,l}$ are known for two sources, whereas just Ω_1 and Ω_2 are needed for the relays. The noise vectors $\mathbf{n}_{j,l} = [n_{j,l}^{(1)}, n_{j,l}^{(2)}]^T$ are assumed to be independent identically distributed (*i.i.d*) zero-mean complex Gaussian random variables with $\mathbb{E}[n_{j,l}^{(i)}] = \sigma^2$, $i = 1, 2$. Let the $4N$ by one vector $\mathbf{r}_j = [\mathbf{r}_{j,1}^T, \mathbf{r}_{j,2}^T, \cdots, \mathbf{r}_{j,2N}^T]^T$, we have

$$\mathbf{r}_j = \sqrt{P_1}\mathbf{H}_{1,j}\mathbf{s}_1 + \sqrt{P_2}\mathbf{H}_{2,j}\mathbf{s}_2 + \mathbf{n}_j, \tag{2}$$

where $\mathbf{H}_{k,j} = \mathbf{I}_{2N} \otimes \mathbf{h}_{k,j}$ and $\mathbf{n}_j = [\mathbf{n}_{j,1}^T, \mathbf{n}_{j,2}^T, \cdots, \mathbf{n}_{j,2N}^T]^T$.

In the second phase, R_j combines the $4N$ received signals from two antennas into new symbols and transmit them with power P_r in consecutive $2N$ time slots simultaneously. Amplify-and-forward (AF) relaying protocol is used in all the relays and a power factor is used to constraint the relay's transmission power. By using $\mathbf{y}_k = \left[y_{k,1}, y_{k,2}, \cdots, y_{k,2N}\right]^T$ as the received signals of T_k during the second phase, we have

$$\mathbf{y}_k = \sum_{j=1}^{N} \left(h_{k,2j-1}\mathbf{t}_j^{(1)} + h_{k,2j}\mathbf{t}_j^{(2)}\right) + \boldsymbol{\eta}_k, \tag{3}$$

where $\mathbf{t}_j^{(i)} = [t_{j,1}^{(i)}, t_{j,2}^{(i)}, \cdots, t_{j,2N}^{(i)}]^T$, and $t_{j,l}^{(i)}$ denotes the signal transmitted in the l-th time slot from the i-th antenna of the R_j node. Furthermore, $\boldsymbol{\eta}_k = [\eta_{k,1}, \eta_{k,2}, \cdots, \eta_{k,2N}]^T$ is a $2N$ by one complex Gaussian noise vector received by T_k with zero mean and covariance matrix $\sigma^2\mathbf{I}_{2N}$.

Define the transmitted signal at the l-th time slot of the j-th relay as $\mathbf{t}_{j,l}$, i.e., $\mathbf{t}_{j,l} = [t_{j,l}^{(1)},\ t_{j,l}^{(2)}]^T$, which is generated by linearly combining $\mathbf{r}_j$ and its conjugate $\mathbf{r}_j^*$ as

$$\mathbf{t}_{j,l} = \sqrt{\beta}(\mathbf{A}_{j,l}\mathbf{r}_j + \mathbf{B}_{j,l}\mathbf{r}_j^*), \tag{4}$$

where $(\mathbf{A}_{j,l})_{2\times 4N}$ and $(\mathbf{B}_{j,l})_{2\times 4N}$ are the coefficient matrices adopted for the l-th time slot of the j-th relay. Eq. (4) is normalized to make the average transmitted power per transmission used at every antenna of every relay be P_r. β denotes the AF factor which ensures that the transmitted relay signals obey this energy constraint. Without loss of generality, we further assume that the total power per symbol transmission used in the whole network is fixed as 1, i.e., $2NP_r + P_1 + P_2 = 1$.

A. General System Model for Detection and Pairwise-Error-Probability Analysis

In this section, we first present two equivalent expressions for the received signals at user nodes and then present the PEP performance analysis. The received signals by T_k at the l-th time slot of the second phase can be written as

$$y_{k,l} = \sqrt{\beta}\sum_{j=1}^{N}(\mathbf{h}_{k,j}^T\mathbf{A}_{j,l}\mathbf{r}_j + \mathbf{h}_{k,j}^T\mathbf{B}_{j,l}\mathbf{r}_j^*) + \eta_{k,l}. \tag{5}$$

Without loss of generality, we only present the equivalent signal model at T_1. Substituting $\mathbf{r}_j$ from (2) into (5) and then eliminating the self-interference caused by $\mathbf{s}_1$, we obtain the l-th received signal at T_1 as

$$z_{1,l} = \sqrt{P_2\beta}\sum_{j=1}^{N}\left(\mathbf{h}_{1,j}^T\mathbf{A}_{j,l}\mathbf{H}_{2,j}\mathbf{s}_2 + \mathbf{h}_{1,j}^T\mathbf{B}_{j,l}\mathbf{H}_{2,j}^*\mathbf{s}_2^*\right) + \xi_{1,l}, \tag{6}$$

where $\xi_{1,l} = \sqrt{\beta}\sum_{j=1}^{N}(\mathbf{h}_{1,j}^T\mathbf{A}_{j,l}\mathbf{n}_j + \mathbf{h}_{1,j}^T\mathbf{B}_{j,l}\mathbf{n}_j^*) + \eta_{1,l}$. Then, letting $\mathbf{z}_1 = [z_{1,1}, z_{1,2}, \cdots, z_{1,2N}]^T$, we get

$$\mathbf{z}_1 = \sqrt{P_2\beta}\mathbf{A}\mathbb{H}_1\mathbf{H}_2\mathbf{s}_2 + \sqrt{P_2\beta}\mathbf{B}\mathbb{H}_1\mathbf{H}_2^*\mathbf{s}_2^* + \boldsymbol{\xi}_1, \tag{7}$$

where $(\mathbb{H}_1)_{8N^2\times 4N^2} = \mathrm{diag}(\mathbf{I}_{4N} \otimes \mathbf{h}_{1,1}, \cdots, \mathbf{I}_{4N} \otimes \mathbf{h}_{1,N})$, $(\mathbf{H}_2)_{4N^2\times 2N} = [\mathbf{H}_{2,1}^T, \mathbf{H}_{2,2}^T, \cdots, \mathbf{H}_{2,N}^T]^T$, $\mathbf{A}_{2N\times 8N^2} = [\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_{2N}]^T$ and $\mathbf{B}_{2N\times 8N^2} = [\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_{2N}]^T$, in which $(\mathbf{a}_l)_{8N^2\times 1} = [\mathrm{vec}^T(\mathbf{A}_{1,l}), \mathrm{vec}^T(\mathbf{A}_{2,l}), \cdots, \mathrm{vec}^T(\mathbf{A}_{N,l})]^T$, $(\mathbf{b}_l)_{8N^2\times 1} = [\mathrm{vec}^T(\mathbf{B}_{1,l}), \mathrm{vec}^T(\mathbf{B}_{2,l}), \cdots, \mathrm{vec}^T(\mathbf{B}_{N,l})]^T$. In addition, the combined noise $\boldsymbol{\xi}_1$ can be written as $\boldsymbol{\xi}_1 = \sqrt{\beta}\mathbf{A}\mathbb{H}_1\mathbf{n} + \sqrt{\beta}\mathbf{B}\mathbb{H}_1\mathbf{n}^* + \boldsymbol{\eta}_1$, in which $\mathbf{n} = [\mathbf{n}_1^T, \mathbf{n}_2^T, \cdots, \mathbf{n}_N^T]^T$. We can also rewrite Eq. (7) as

$$\mathbf{z}_1 = \sqrt{P_2\beta}\mathbf{A}\mathbb{S}_2\check{\mathbf{H}}_1\mathbf{h}_2 + \sqrt{P_2\beta}\mathbf{B}\mathbb{S}_2^*\check{\mathbf{H}}_1\mathbf{h}_2^* + \boldsymbol{\xi}_1, \tag{8}$$

where $(\mathbb{S}_2)_{8N^2\times 4N} = \mathbf{I}_N \otimes (\mathbf{s}_2 \otimes \mathbf{I}_4)$, $(\check{\mathbf{H}}_1)_{4N\times 2N} = \mathrm{diag}(\mathbf{I}_2 \otimes \mathbf{h}_{1,1}, \mathbf{I}_2 \otimes \mathbf{h}_{1,2}, \cdots, \mathbf{I}_2 \otimes \mathbf{h}_{1,N})$ and $\mathbf{h}_2 = [\mathbf{h}_{2,1}^T, \mathbf{h}_{2,2}^T, \cdots, \mathbf{h}_{2,N}^T]^T$. For the convenience of PEP analysis, (8) can also be expressed as a real-valued form by vectorizing the complex vectors in terms of their real and imaginary parts as follows

$$\bar{\mathbf{z}}_1 = \sqrt{P_2\beta}\mathbf{K}_{2N}^H\mathbf{M}\mathbf{K}_{2N}\bar{\mathbf{h}}_2 + \bar{\boldsymbol{\xi}}_1, \tag{9}$$

where the noise vector

$$\bar{\boldsymbol{\xi}}_1 = \sqrt{\beta}\mathbf{K}_{2N}^H\mathbf{N}\mathbf{K}_{4N^2}\bar{\mathbf{n}} + \bar{\boldsymbol{\eta}}_1, \tag{10}$$

$\bar{\mathbf{z}}_1 = \mathcal{R}(\mathbf{z}_1)$, $\bar{\mathbf{h}}_2 = \mathcal{R}(\mathbf{h}_2)$, $\bar{\boldsymbol{\xi}}_1 = \mathcal{R}(\boldsymbol{\xi}_1)$, $\bar{\mathbf{n}} = \mathcal{R}(\mathbf{n})$, $\bar{\boldsymbol{\eta}}_1 = \mathcal{R}(\boldsymbol{\eta}_1)$, $\mathbf{K}_n = \frac{1}{\sqrt{2}}\begin{bmatrix}\mathbf{I}_n & j\mathbf{I}_n \\ \mathbf{I}_n & -j\mathbf{I}_n\end{bmatrix}$, $\mathbf{N} = \begin{bmatrix}\mathbf{A}\mathbb{H}_1 & \mathbf{B}\mathbb{H}_1 \\ \mathbf{B}^*\mathbb{H}_1^* & \mathbf{A}^*\mathbb{H}_1^*\end{bmatrix}$, and $\mathbf{M} = \begin{bmatrix}\mathbf{A}\mathbb{S}_2\check{\mathbf{H}}_1 & \mathbf{B}\mathbb{S}_2^*\check{\mathbf{H}}_1 \\ \mathbf{B}^*\mathbb{S}_2\check{\mathbf{H}}_1^* & \mathbf{A}^*\mathbb{S}_2^*\check{\mathbf{H}}_1^*\end{bmatrix}$.

B. PEP Analysis for the General Network

Eq. (9) can be written as a more common form for simplicity, i.e.,

$$\bar{\mathbf{z}}_1 = \sqrt{P_2\beta}\mathcal{X}(\mathbf{h}_1, \mathbf{s}_2)\bar{\mathbf{h}}_2 + \bar{\boldsymbol{\xi}}_1, \tag{11}$$

where $\mathcal{X}(\mathbf{h}_1, \mathbf{s}_2) = \mathbf{K}_{2N}^H\mathbf{M}\mathbf{K}_{2N}$ and it is a $4N \times 4N$ signal matrix. According to (10), for any fixed $\mathbf{h}_1$, $\bar{\boldsymbol{\xi}}_1$ is an independently circularly-symmetric complex Gaussian noise with the zero mean and covariance matrix $\Sigma = \frac{\sigma^2\beta}{2}\mathbf{K}_{2N}^H\mathbf{N}\mathbf{N}^H\mathbf{K}_{2N} + \frac{\sigma^2}{2}\mathbf{I}_{4N}$. Thus, given a channel realization, the probability $P\left(\mathbf{s}_2 \to \mathbf{s}_2'|\mathbf{h}_1, \mathbf{h}_2\right)$ of transmitting $\mathbf{s}_2$ and deciding in favor of $\mathbf{s}_2' \neq \mathbf{s}_2$ with the ML detector is given by [44]

$$P\left(\mathbf{s}_2 \to \mathbf{s}_2'|\mathbf{h}_1, \mathbf{h}_2\right) = Q\left(\frac{d(\mathbf{s}_2, \mathbf{s}_2')}{\sqrt{2}}\right), \tag{12}$$

where $d(\mathbf{s}_2, \mathbf{s}_2')$ is defined by $d^2(\mathbf{s}_2, \mathbf{s}_2') = P_2\beta\bar{\mathbf{h}}_2^H\mathcal{X}^H(\mathbf{h}_1, \mathbf{e})\Sigma^{-1}\mathcal{X}(\mathbf{h}_1, \mathbf{e})\bar{\mathbf{h}}_2$ with $\mathbf{e} = \mathbf{s}_2 - \mathbf{s}_2'$, and the Q function $Q(x) = \frac{1}{\pi}\int_0^{\pi/2}\exp\left(-\frac{x^2}{2\sin^2\theta}\right)d\theta$.

Now, taking the average of (12) over the random $4N$ by one vector $\bar{\mathbf{h}}_2$, we have

$$P\left(\mathbf{s}_2 \to \mathbf{s}_2' | \mathbf{h}_1\right) = \frac{1}{\pi}\int_0^{\pi/2} \frac{d\theta}{\det^{\frac{1}{2}}\left(\mathbf{I} + \beta_2(\theta)\, \mathcal{X}^H(\mathbf{h}_1, \mathbf{e})\, \Sigma^{-1} \mathcal{X}(\mathbf{h}_1, \mathbf{e})\right)}, \quad (13)$$

where $\beta_2(\theta) = \left(4\sin^2\theta\right)^{-1} P_2 \Omega_2 \beta$. By taking the expectation of (13) over $\mathbf{h}_1$, we obtain the following theorem, which gives the lower bound of the PEP.

Theorem 1: There exist two constants $C_{k,N}, k = 1, 2$, independent of SNR ρ, such that the PEPs at T_k satisfy

$$P\left(\mathbf{s}_{\bar{k}} \to \mathbf{s}_{\bar{k}}'\right) = \mathbb{E}_{\mathbf{h}_k}\left[P\left(\mathbf{s}_{\bar{k}} \to \mathbf{s}_{\bar{k}}' | \mathbf{h}_k\right)\right] \geq C_{k,N}\, \rho^{-2N} \ln^N \rho, \quad (14)$$

when SNR ρ is large, where $\left(k, \bar{k}\right) \in \{(1,2), (2,1)\}$. ■

Proof: The detailed proof is similar to that in [25].

Remarks: Theorem 1 shows that the lower bound PEP of our considered TWRNs. To achieve this lower bound, the coefficient matrices at relays should be carefully designed. We call one design as the optimal design if its PEP can achieve this lower bound. We have presented two optimal designs for the special cases with $N = 1$ in [24] and $N = 2$ in [25]. For multiple relays, $N > 2$, we will use an upper-lower bound strategy and the induction method to investigate the asymptotic PEP in view of the mathematic difficulty of the conventional direct methods [24], [25].

III. Optimal Distributed Concatenated Recursive Alamouti-Circulant STBCs

In this section, we proposed a feasible DSTBC design, which is verified optimal to achieve the PEP lower bound. Specially, we consider the orthogonal or quasi-orthogonal DSTBC for such a networks with $N = 2^b$ relays, where b is a nonnegative integer. In addition, we use the Alamouti code as the elementary code block as to expect achieving full-diversity and low-complexity detection. With these assumptions, we can find the fact that the equivalent channel at each source node is the so-called recursive Alamouti-circulant (RAC) matrix, and its each block is the product of the two Alamouti channel matrices. Before we present our design, we first give the definition of the RAC matrix.

Definition 1: Let a two-by-two matrix Alamouti code matrix be $\begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix}, a, b \in \mathbb{C}$ We call the Alamouti code matrix RACM_1. Then the $2^n \times 2^n$ RAC matrix set, denoted as RACM_n, is given by

$$\mathrm{RACM}_n = \left\{ \begin{bmatrix} \mathbf{M}_1 & \mathbf{M}_2 \\ \mathbf{M}_2 & \mathbf{M}_1 \end{bmatrix}, \mathbf{M}_\ell \in \mathrm{RACM}_{n-1} \text{ for } \ell = 1, 2 \right\},$$

for any integer n larger than one. ■

From *Definition* 1, we have the following proposition, which is crucial when we analyze the diversity performance of our design.

Proposition 1: A RAC matrix $\mathbf{G}_b \in \mathrm{RACM}_{b+1}$ can be decomposed into

$$\mathbf{G}_b = \mathbf{T}_b^H \mathbf{D}_b \mathbf{T}_b, \quad (15)$$

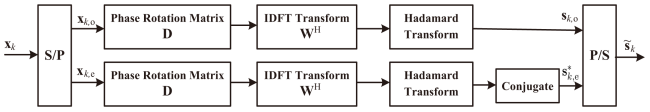

Fig. 2. Diagram illustration of the signal design at T_k for the proposed distributed concatenated recursive Alamouti-circulant STBC.

where $\mathbf{D}_b$ is a block diagonal matrix, i.e., $\mathbf{D}_b = \sqrt{2^b}\mathrm{diag}(\mathbf{T}_{b,1}\mathbf{g}_b, \mathbf{T}_{b,2}\mathbf{g}_b, \cdots, \mathbf{T}_{b,N}\mathbf{g}_b)$, $\mathbf{T}_{b,j}$ denotes a $2 \times 2^{b+1}$ matrix consisting of the $(2j-1)$-th and $(2j)$-th rows of $\mathbf{T}_b$, and $\mathbf{g}_b$ is a $2^{b+1} \times 2$ matrix consisting of the first two columns of $\mathbf{G}_b$. Here, $\mathbf{T}_b$ is a unitary block Hadamard matrix, which is defined as $\mathbf{T}_b = \frac{1}{\sqrt{2^b}}\mathrm{Hadamard}(2^b) \otimes \mathbf{I}_2$. ■

Proposition 1 can be proved by using the induction on b. See Appendix B for more details.

A. Description of the Proposed Design

Let $x_{k,l}$, $l = 1, \cdots, 2N$, be the signals generated from a standard M-QAM constellation $\mathcal{Q}$ with unit symbol energy. At T_k, $2N$ complex constellation symbols $\mathbf{x}_k$ are divided into two groups, i.e., $\mathbf{x}_{k,\mathrm{o}} = [x_{k,1}, x_{k,3}, \cdots, x_{k,2N-1}]^T$ and $\mathbf{x}_{k,\mathrm{e}} = [x_{k,2}, x_{k,4}, \cdots, x_{k,2N}]^T$. As shown in Fig. 2, to achieve full diversity, both groups are precoded individually by the angle rotation transformation $\mathbf{D}$, the inverse discrete Fourier transform (IDFT) transformation $\mathbf{W}_N^H$, and the Hadamard transformation. Then $\mathbf{s}_{k,\mathrm{o}}$ and $\mathbf{s}_{k,\mathrm{e}}^*$ are combined into the transmitted signal $\mathbf{s}_k$. The whole process can be expressed as $\mathbf{s}_{k,\mathrm{o}} = \mathbf{P}\mathbf{x}_{k,\mathrm{o}}$ and $\mathbf{s}_{k,\mathrm{e}}^* = \mathbf{P}^*\mathbf{x}_{k,\mathrm{e}}^*$, where $\mathbf{P} = \frac{1}{\sqrt{N}}\mathrm{Hadamard}(N)\mathbf{W}_N^H\mathbf{D}$. Here, $\mathbf{W}_N$ is a DFT matrix, i.e., $\mathbf{W}_N(p,q) = \frac{1}{\sqrt{N}}e^{-j2\pi pq/N}$, $p, q = 1, \cdots, N$. To ensure the full rank of the difference matrix and maximize its determinant, $N-1$ rotation angles are required, i.e., $\mathbf{D} = \mathrm{diag}(1, e^{j\frac{\pi}{2N}}, \cdots, e^{j(N-1)\frac{\pi}{2N}})$. By combining $\mathbf{s}_{k,\mathrm{o}}$ and $\mathbf{s}_{k,\mathrm{e}}^*$, we can obtain

$$\widetilde{\mathbf{s}}_k = \mathbf{E}\mathrm{diag}(\mathbf{P}, \mathbf{P}^*)\mathbf{E}^H\widetilde{\mathbf{x}}_k, \quad (16)$$

where $\mathbf{E}$ is a $2N \times 2N$ elementary permutation matrix which permutes $[\mathbf{x}_{k,\mathrm{o}}^T, \mathbf{x}_{k,\mathrm{e}}^T]^T$ into $\mathbf{x}_k$. In addition, $\widetilde{\mathbf{x}}_k = [x_{k,1}, x_{k,2}^*, \cdots, x_{k,2N-1}, x_{k,2N}^*]^T$ and $\widetilde{\mathbf{s}}_k = [s_{k,1}, s_{k,2}^*, \cdots, s_{k,2N-1}, s_{k,2N}^*]^T$. In the first consecutive $2N$ time slots, $\widetilde{\mathbf{s}}_1$ and $\widetilde{\mathbf{s}}_2$ are transmitted to the relays simultaneously. Each relay then generates the broadcasting signals by linearly combining the received $4N$ signals from its two antennas and broadcasts them in the following consecutive $2N$ time slots.

Let $\boldsymbol{\eta}_k = [\eta_{k,1}, \eta_{k,2}, \cdots, \eta_{k,2N}]^T$ still be a $2N$ by one complex Gaussian noise vector received by T_k with zero mean and covariance matrix $\sigma^2\mathbf{I}_{2N}$, the received signals at $\mathrm{T}_k, k = 1, 2$ are then denoted as

$$\mathbf{y}_k = \sqrt{\beta}\mathbf{R}^T\mathbf{h}_k + \boldsymbol{\eta}_k, \quad (17)$$

where $\mathbf{R}$ is the generated transmission signal matrix by the $N = 2^b$ relays. Recall (3), $\mathbf{R}$ is generated in a way as

$$\mathbf{R} = (\mathbf{R}_{j,i})_{N\times N} = \begin{bmatrix} \mathbf{t}_1^{(1)} & \mathbf{t}_1^{(2)} & \cdots & \mathbf{t}_N^{(1)} & \mathbf{t}_N^{(2)} \end{bmatrix}^T = \begin{bmatrix} \mathbf{R}_{1,1} & \mathbf{R}_{1,2} & \cdots & \mathbf{R}_{1,N} \\ \mathbf{R}_{2,1} & \mathbf{R}_{2,2} & \cdots & \mathbf{R}_{2,N} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{R}_{N,1} & \mathbf{R}_{N,2} & \cdots & \mathbf{R}_{N,N} \end{bmatrix}, \quad (18)$$

in which

$$\mathbf{R}_{j,i} = \begin{bmatrix}\mathbf{t}_{j,2i-1} & \mathbf{t}_{j,2i}\end{bmatrix} = \begin{bmatrix} r^{(1)}_{j,L_{j,i}} + r^{(2)*}_{j,L_{j,i}+1} & r^{(1)}_{j,L_{j,i}+1} - r^{(2)*}_{j,L_{j,i}} \\ r^{(2)}_{j,L_{j,i}} - r^{(1)*}_{j,L_{j,i}+1} & r^{(1)*}_{j,L_{j,i}} + r^{(2)}_{j,L_{j,i}+1}\end{bmatrix}.$$

Here, $L_{j,i}$ denotes the received signal index during the $(2i-1)$-th time slot of the j-th relay, $j, i \in \{1, \cdots, N\}$. If let $\mathbf{L}_b = (L_{j,i})_{N\times N}$ denote the index matrix, then for our design, $\mathbf{L}_b$ has the following structure.

Definition 2: If $b = 0$, $\mathbf{L}_0 = 1$, otherwise if $b > 0$, the $2^b \times 2^b$ matrix $\mathbf{L}_b$ is defined as

$$\mathbf{L}_b = \begin{bmatrix}\mathbf{L}_{b-1} & 2^b\mathbf{1}_{b-1} + \mathbf{L}_{b-1} \\ 2^b\mathbf{1}_{b-1} + \mathbf{L}_{b-1} & \mathbf{L}_{b-1}\end{bmatrix},$$

where $\mathbf{1}_\ell$ is a $2^\ell \times 2^\ell$ matrix of all ones, and b is a nonnegative integer. ■

It should be pointed out that since the average transmission energy per symbol used at each antenna of the relay nodes is normalized to be P_r, the AF amplifier β must be accordingly chosen in such a way that $\beta = \frac{P_r}{2(\Omega_1 P_1 + \Omega_2 P_2 + \sigma^2)} \approx \frac{P_r}{2(\Omega_1 P_1 + \Omega_2 P_2)}$ from (17). In addition, according to the general definition given by (4), i.e., $\mathbf{t}_{j,l} = \sqrt{\beta}(\mathbf{A}_{j,l}\mathbf{r}_j + \mathbf{B}_{j,l}\mathbf{r}^*_j)$, we get

$$\begin{cases}\mathbf{A}_{j,2i-1}(1, 2L_{j,i} - 1) = 1 \\ \mathbf{A}_{j,2i-1}(2, 2L_{j,i}) = 1 \\ \mathbf{A}_{j,2i}(1, 2L_{j,i} + 1) = 1 \\ \mathbf{A}_{j,2i}(2, 2L_{j,i} + 2) = 1,\end{cases}$$

$$\begin{cases}\mathbf{B}_{j,2i-1}(1, 2L_{j,i} + 2) = 1 \\ \mathbf{B}_{j,2i-1}(2, 2L_{j,i} + 1) = -1 \\ \mathbf{B}_{j,2i}(1, 2L_{j,i}) = -1 \\ \mathbf{B}_{j,2i}(2, 2L_{j,i} - 1) = 1,\end{cases}$$

where all the remaining elements of $\mathbf{A}$ and $\mathbf{B}$ are zero. It can be seen that

1) $\mathbf{A}_{j,l}\mathbf{A}^T_{j,l'} = \mathbf{O}$ and $\mathbf{B}_{j,l}\mathbf{B}^T_{j,l'} = \mathbf{O}$ for $l \neq l'$;
2) $\mathbf{A}_{j,l}\mathbf{A}^T_{j,l} = \mathbf{I}_2$ and $\mathbf{B}_{j,l}\mathbf{B}^T_{j,l} = \mathbf{I}_2$;
3) $\mathbf{A}_{j,2i-1}\mathbf{B}^T_{j,2i} = \mathbf{A}_{j,2i}\mathbf{B}^T_{j,2i-1} = \begin{bmatrix}0 & 1 \\ -1 & 0\end{bmatrix}$ for $i = 1, \cdots, N$, otherwise $\mathbf{A}_{j,l}\mathbf{B}^T_{j,l'} = \mathbf{O}$.

Compared with [25], we can see that our proposed design is more general for $N = 2^b$. However, the work in [25] is just a special case for $N = 2$ which can not be directly extended to our scenario without the precoding precessing and the concept of recursive Alamouti-circulant (RAC) matrix.

B. Equivalent Received Signal Model for Detection

We assume the self-interference caused by the transmitted signal $\mathbf{s}_k$ at T_k can be cancelled ideally. By using some basic mathematic transformations, we can rewrite (17) into

$$\widetilde{\mathbf{z}}_1 = \sqrt{P_2\beta}\mathbf{G}_b\mathbf{s}_2 + \boldsymbol{\epsilon}_1, \tag{19a}$$
$$\widetilde{\mathbf{z}}_2 = \sqrt{P_1\beta}\mathbf{G}^T_b\mathbf{s}_1 + \boldsymbol{\epsilon}_2, \tag{19b}$$

where $\widetilde{\mathbf{z}}_k = [z_{k,1}, z^*_{k,2}, z_{k,3}, z^*_{k,4}, \cdots, z_{k,2N-1}, z^*_{k,2N}]^T$ denotes the received equivalent signal vector at $\mathrm{T_k}$ after self-interference cancelling from $\mathbf{y}_k$ and taking the conjugation for the terms with even indices. Specifically, $\mathbf{G}_b$ is a $2N \times 2N$ RACM, in which the first two row is $[\mathcal{H}_{1,1}\mathcal{H}^T_{2,1}\ \mathcal{H}_{1,2}\mathcal{H}^T_{2,2}\ \cdots\ \mathcal{H}_{1,N}\mathcal{H}^T_{2,N}]$. Furthermore,

$$\boldsymbol{\epsilon}_k = \sqrt{\beta}\begin{bmatrix}\sum_{j=1}^N \mathcal{H}_{k,j}\mathbf{w}_{j,1} \\ \sum_{j=1}^N \mathcal{H}_{k,j}\mathbf{w}_{j,2} \\ \cdots \\ \sum_{j=1}^N \mathcal{H}_{k,j}\mathbf{w}_{j,N}\end{bmatrix} + \begin{bmatrix}\widetilde{\boldsymbol{\eta}}_{k,1} \\ \widetilde{\boldsymbol{\eta}}_{k,2} \\ \cdots \\ \widetilde{\boldsymbol{\eta}}_{k,N}\end{bmatrix},$$

where $\widetilde{\boldsymbol{\eta}}_{k,j} = [\eta_{k,2j-1}, \eta^*_{k,2j}]^T$, $\mathbf{w}_{j,i} = [w^{(1)}_{j,i}, w^{(2)}_{j,i}]^T = [n^{(1)}_{j,L_{j,i}} + n^{(2)*}_{j,L_{j,i}+1}, n^{(1)*}_{j,L_{j,i}+1} - n^{(2)}_{j,L_{j,i}}]^T$ and $\mathcal{H}_{k,j} = \begin{bmatrix}h_{k,2j-1} & -h_{k,2j} \\ h^*_{k,2j} & h^*_{k,2j-1}\end{bmatrix}$. Due to $\mathbf{s}_k = \mathbf{E}\mathrm{diag}(\mathbf{P}, \mathbf{P})\mathbf{E}^H\mathbf{x}_k$, (19) can be also written as

$$\widetilde{\mathbf{z}}_1 = \sqrt{P_2\beta}\mathbf{G}_b\mathbf{E}\mathrm{diag}(\mathbf{P}, \mathbf{P})\mathbf{E}^H\mathbf{x}_2 + \boldsymbol{\epsilon}_1, \tag{20a}$$
$$\widetilde{\mathbf{z}}_2 = \sqrt{P_1\beta}\mathbf{G}^T_b\mathbf{E}\mathrm{diag}(\mathbf{P}, \mathbf{P})\mathbf{E}^H\mathbf{x}_1 + \boldsymbol{\epsilon}_2. \tag{20b}$$

C. Equivalent Received Signal Model for PEP Analysis

Since (19) is not easy to be used to derive the theoretic PEP performance, in this part, we present an equivalent formulation for the diversity analysis in the next section. For simplicity and the symmetric property of two users, we still focus on T_1 only.

By using the *Proposition* 1, (19a) can be written as

$$\mathbf{T}_b\widetilde{\mathbf{z}}_1 = \sqrt{P_2\beta}\mathbf{D}_b\boldsymbol{\chi}_2 + \boldsymbol{\zeta}_1, \tag{21}$$

where $\mathbf{D}_b = \mathbf{T}_b\mathbf{G}_b\mathbf{T}^H_b$, $\boldsymbol{\chi}_k = \mathbf{T}_b\mathbf{s}_k = [\chi_{k,1}, \chi_{k,2}, \cdots, \chi_{k,2N}]^T$ and $\boldsymbol{\zeta}_k = \mathbf{T}_b\boldsymbol{\epsilon}_k = [\zeta_{k,1}, \zeta_{k,2}, \cdots, \zeta_{k,2N}]^T$. Conjugating the terms with even indices and using some basic mathematical transformations, we have

$$\mathbf{T}_b\mathbf{z}_1 = \sqrt{2P_2\beta}\mathcal{S}_2\mathbf{F}_1\mathbf{K}_{2N}\bar{\mathbf{h}}_2 + \widetilde{\boldsymbol{\zeta}}_1, \tag{22}$$

where $\mathcal{S}_k = \mathrm{diag}(\mathcal{S}_{k,1}, \mathcal{S}_{k,2}, \cdots, \mathcal{S}_{k,N})$ with $\mathcal{S}_{k,j} = \begin{bmatrix}\chi_{k,2j-1} & -\chi_{k,2j} \\ \chi^*_{k,2j} & \chi^*_{k,2j-1}\end{bmatrix}$, $\bar{\mathbf{h}}_k = \mathcal{R}(\mathbf{h}_k)$ and the matrix $\mathbf{F}_k$ is given by

$$\mathbf{F}_k = \begin{bmatrix}\mathbf{F}_{k,1} & \mathbf{F}_{k,2} \\ \mathbf{F}_{k,3} & \mathbf{F}_{k,4} \\ \cdots & \cdots \\ \mathbf{F}_{k,2N-1} & \mathbf{F}_{k,2N}\end{bmatrix},$$

in which $\mathbf{F}_{k,2l} = \begin{bmatrix}\mathbf{d}_{2l} \odot \mathbf{h}^T_k \\ \mathbf{0}\end{bmatrix}$ and $\mathbf{F}_{k,2l-1} = \begin{bmatrix}\mathbf{0} \\ \mathbf{d}_{2l-1} \odot \widehat{\mathbf{h}}^T_k\end{bmatrix}$, $l = 1, \cdots, N$. In addition, $\mathbf{d}_l$ denotes the l-th row of the Hadamard matrix of order $2N$ and $\widehat{\mathbf{h}}_k = [h_{k,2}, h_{k,1}, \cdots, h_{k,2N}, h_{k,2N-1}]^T$. Notice that the noise vector $\widetilde{\boldsymbol{\zeta}}_k$ can be expressed as

$$\widetilde{\boldsymbol{\zeta}}_k = \sqrt{\beta}\mathbf{T}_b\mathcal{W}\mathbf{h}_k + \mathbf{T}_b\boldsymbol{\eta}_k, \tag{23}$$

where $\mathcal{W} = (\mathcal{W}_{ji})_{N\times N}$ and $\mathcal{W}_{ji} = \begin{bmatrix}w^{(1)}_{j,i} & -w^{(2)}_{j,i} \\ w^{(2)*}_{j,i} & w^{(1)*}_{j,i}\end{bmatrix}$. Due to the fact that $\mathcal{W}_{j_1 i_1}$ is independent with $\mathcal{W}_{j_2 i_2}$ for any possible $\{j_1, i_1\} \in \{j, i\} \neq \{j_2, i_2\} \in \{j, i\}$, we have

$$\Sigma_k = \mathbf{E}[\widetilde{\boldsymbol{\zeta}}_k\widetilde{\boldsymbol{\zeta}}^H_k] = \beta\mathbb{E}[\mathbf{T}_b\mathcal{W}\mathbf{h}_k\mathbf{h}^H_k\mathcal{W}^H\mathbf{T}_b] + \sigma^2\mathbf{I}_{2N} = (2\beta\|\mathbf{h}_k\|^2 + 1)\sigma^2\mathbf{I}_{2N}. \tag{24}$$

Thus, (22) can be transformed into the following expression:

$$\bar{\mathbf{z}}_{\mathrm{T},1} = \sqrt{P_2\beta}\mathbf{K}_{2N}^H \mathrm{diag}(\mathcal{S}_2, \mathcal{S}_2^*)\bar{\mathbf{F}}_1\mathbf{K}_{2N}\bar{\mathbf{h}}_2 + \bar{\boldsymbol{\zeta}}_1, \quad (25)$$

where $\bar{\mathbf{z}}_{\mathrm{T},k} = \mathcal{R}(\mathbf{T}_b\mathbf{z}_k)$, $\bar{\boldsymbol{\zeta}}_k = \mathcal{R}(\widetilde{\boldsymbol{\zeta}}_k)$, and $\bar{\mathbf{F}}_k = [\mathbf{F}_k; \mathbf{F}_k^*\mathbf{B}]$ in which $\mathbf{B} = \begin{bmatrix} \mathbf{O} & \mathbf{I}_{2N} \\ \mathbf{I}_{2N} & \mathbf{O} \end{bmatrix}$.

D. Comments and Examples

1) From (20), we can see that the equivalent channel matrix $\mathbf{G}_b$ and its transpose are the specific RAC matrices. It should be mentioned that the structure of the channel matrix $\mathbf{G}_b$ is mimic the structure of the rate one STBC proposed in [45]. However, the major difference here is that each 2×2 block entry of $\mathbf{G}_b$ is the product of two Alamouti matrices. It is for this reason that we call our code as a distributed recursive block-circulant STBC.
2) Power loading among source nodes and relay nodes significantly affects the overall performance of the whole relay system [11], [31]. One solution of the optimal power allocation can be obtained by maximizing the received SNR of the weak user link, which is given in the following proposition.

   Proposition 2: The optimal power loading to maximize the average received SNR of the weak user link is determined as follows:

$$P_r = \frac{1}{4N},$$
$$P_1 = \frac{\sqrt{\Omega_2}}{2(\sqrt{\Omega_1}+\sqrt{\Omega_2})}, \quad P_2 = \frac{\sqrt{\Omega_1}}{2(\sqrt{\Omega_1}+\sqrt{\Omega_2})}.$$

   ■

   The proof of Proposition 2 follows that we used in [25], so it is omitted. It reveals that the optimal total power assigned to all relays is half of the total network power, regardless of the channel variances.
3) It is known that when CSI is perfectly available at the source nodes, the optimal detector for estimation of the transmitted signals is the ML detector, which is to solve the following two optimization problems:

$$\mathbf{x}_2' = \arg\min_{\mathbf{x}_2} \|\widetilde{\mathbf{z}}_1 - \sqrt{P_2\beta}\mathbf{G}_b\mathbf{E}\mathrm{diag}(\mathbf{P},\mathbf{P})\mathbf{E}^H\mathbf{x}_2\|^2, \quad (26a)$$

$$\mathbf{x}_1' = \arg\min_{\mathbf{x}_1} \|\widetilde{\mathbf{z}}_2 - \sqrt{P_1\beta}\mathbf{G}_b^T\mathbf{E}\mathrm{diag}(\mathbf{P},\mathbf{P})\mathbf{E}^H\mathbf{x}_1\|^2. \quad (26b)$$

   Due to the *Proposition* 1, the minimization of (26a) is equivalent to minimizing the following two expressions individually:

$$\mathbf{x}_{2,\mathrm{o}}' = \arg\min_{\mathbf{s}_{2,\mathrm{o}}} \|\widetilde{\mathbf{z}}_1 - \sqrt{P_2\beta}\mathbf{G}_{b,\mathrm{o}}\mathbf{P}\mathbf{x}_{2,\mathrm{o}}\|^2, \quad (27a)$$

$$\mathbf{x}_{2,\mathrm{e}}' = \arg\min_{\mathbf{s}_{2,\mathrm{e}}} \|\widetilde{\mathbf{z}}_1 - \sqrt{P_2\beta}\mathbf{G}_{b,\mathrm{e}}\mathbf{P}\mathbf{x}_{2,\mathrm{e}}\|^2, \quad (27b)$$

   where $\mathbf{G}_{b,\mathrm{o}}$ and $\mathbf{G}_{b,\mathrm{e}}$ denote the $2N \times N$ matrices including only odd columns and even columns of $\mathbf{G}_b$ respectively.
4) It is very interesting to see that (25) is equivalent to (22) in a sense that is given by the following lemma.

Lemma 1: The following two determinants are equal

$$\det\left(\mathbf{I}_{4N} + \alpha_k\bar{\mathbf{F}}_{\bar{k}}^H\mathrm{diag}(\mathcal{S}_k^H, \mathcal{S}_k^T)\mathrm{diag}(\mathcal{S}_k, \mathcal{S}_k^*)\bar{\mathbf{F}}_{\bar{k}}\right)^{\frac{1}{2}} = \det\left(\mathbf{I}_{2N} + \alpha_k\mathbf{F}_{\bar{k}}\mathbf{F}_{\bar{k}}^H\mathcal{S}_k^H\mathcal{S}_k\right), \quad (28)$$

where α_k is a constant independent with $\mathbf{s}_k$ and $\mathbf{h}_k$, $k = 1, 2$. ■

The proof of Lemma 1 can be completed by using Sylvester's determinant theorem [46] and the fact that $\mathbf{F}_{k,2m}\mathbf{F}_{k,2n-1}^H = \mathbf{O}$, $m, n = 1, \cdots, N$.

For better understanding of our design, we consider the full-rate distributed space-time transmission schemes and further present three typical examples in the following.

Example 1: Consider the TWRN with single dual-antenna relay, i.e., $b = 0$, we have $\mathbf{A}_1 = (\mathbf{I}_2, \mathbf{O}_2)$, $\mathbf{A}_2 = (\mathbf{O}_2, \mathbf{I}_2)$, $\mathbf{B}_1 = (\mathbf{O}_2, \mathbf{J}_2)$, and $\mathbf{B}_2 = (-\mathbf{J}_2, \mathbf{O}_2)$, where $\mathbf{J}_2 = [0,\ 1; -1,\ 0]$. These lead to $\mathbf{G}_0 = \mathcal{H}_{1,1}\mathcal{H}_{2,1}^T$, $\mathbf{E} = \mathbf{I}_2$, and $\mathbf{P}$ turns into 1. Thus, we get $\mathbf{s}_k = \mathbf{x}_k$, which means the transmitted signals $\mathbf{s}_k$ belong to the constellation set $\mathcal{Q}$.

Example 2: For the TWRN with 2 dual-antenna relays, i.e., $b = 1$, we have the coefficients as $\mathbf{A}_{1,1} = \mathbf{A}_{2,3} = (\mathbf{I}_2, \mathbf{O}_2, \mathbf{O}_2, \mathbf{O}_2)$, $\mathbf{A}_{1,2} = \mathbf{A}_{2,4} = (\mathbf{O}_2, \mathbf{I}_2, \mathbf{O}_2, \mathbf{O}_2)$, $\mathbf{A}_{1,3} = \mathbf{A}_{2,1} = (\mathbf{O}_2, \mathbf{O}_2, \mathbf{I}_2, \mathbf{O}_2)$, $\mathbf{A}_{1,4} = \mathbf{A}_{2,2} = (\mathbf{O}_2, \mathbf{O}_2, \mathbf{O}_2, \mathbf{I}_2)$, $\mathbf{B}_{1,1} = \mathbf{B}_{2,3} = (\mathbf{O}_2, \mathbf{J}_2, \mathbf{O}_2, \mathbf{O}_2)$, $\mathbf{B}_{1,2} = \mathbf{B}_{2,4} = (-\mathbf{J}_2, \mathbf{O}_2, \mathbf{O}_2, \mathbf{O}_2)$, $\mathbf{B}_{1,3} = \mathbf{B}_{2,1} = (\mathbf{O}_2, \mathbf{O}_2, \mathbf{O}_2, \mathbf{J}_2)$, and $\mathbf{B}_{1,4} = \mathbf{B}_{2,2} = (\mathbf{O}_2, \mathbf{O}_2, -\mathbf{J}_2, \mathbf{O}_2)$. Hence, we have

$$\mathbf{G}_1 = \begin{bmatrix} \mathcal{H}_{1,1}\mathcal{H}_{2,1}^T & \mathcal{H}_{1,2}\mathcal{H}_{2,2}^T \\ \mathcal{H}_{1,2}\mathcal{H}_{2,2}^T & \mathcal{H}_{1,1}\mathcal{H}_{2,1}^T \end{bmatrix}.$$

In addition, $\mathbf{P} = \mathbf{D} = \mathrm{diag}(1, e^{j\frac{\pi}{4}})$, and $\mathbf{E} = \mathbf{E}^H$ is obtained by exchanging the second row and the third row of $\mathbf{I}_4$. Furthermore, $\mathbf{s}_k = \mathrm{diag}(1, e^{j\frac{\pi}{4}}, 1, e^{j\frac{\pi}{4}})\mathbf{x}_k$.

Example 3: In this case, we present a more complex TWRN with 4 dual-antenna relays, i.e., $b = 2$. Similarly, we have the RAC matrix

$$\mathbf{G}_2 = \begin{bmatrix} \mathcal{H}_{1,1}\mathcal{H}_{2,1}^T & \mathcal{H}_{1,2}\mathcal{H}_{2,2}^T & \mathcal{H}_{1,3}\mathcal{H}_{2,3}^T & \mathcal{H}_{1,4}\mathcal{H}_{2,4}^T \\ \mathcal{H}_{1,2}\mathcal{H}_{2,2}^T & \mathcal{H}_{1,1}\mathcal{H}_{2,1}^T & \mathcal{H}_{1,4}\mathcal{H}_{2,4}^T & \mathcal{H}_{1,3}\mathcal{H}_{2,3}^T \\ \mathcal{H}_{1,3}\mathcal{H}_{2,3}^T & \mathcal{H}_{1,4}\mathcal{H}_{2,4}^T & \mathcal{H}_{1,1}\mathcal{H}_{2,1}^T & \mathcal{H}_{1,2}\mathcal{H}_{2,2}^T \\ \mathcal{H}_{1,4}\mathcal{H}_{2,4}^T & \mathcal{H}_{1,3}\mathcal{H}_{2,3}^T & \mathcal{H}_{1,2}\mathcal{H}_{2,2}^T & \mathcal{H}_{1,1}\mathcal{H}_{2,1}^T \end{bmatrix}$$

and $\mathbf{E} = [\mathbf{I}_8(1,:); \mathbf{I}_8(5,:); \mathbf{I}_8(2,:); \mathbf{I}_8(6,:); \mathbf{I}_8(3,:); \mathbf{I}_8(7,:); \mathbf{I}_8(4,:); \mathbf{I}_8(8,:)]$, in which $\mathbf{I}_8(i,:)$ denotes the i-th row vector of $\mathbf{I}_8$. As a consequence, $\mathbf{s}_{2,\mathrm{o}} = \frac{1}{2}\mathrm{Hadamard}(4)\mathbf{W}_4^H\mathbf{D}\mathbf{x}_{2,\mathrm{o}}$ and $\mathbf{s}_{2,\mathrm{e}} = \frac{1}{2}\mathrm{Hadamard}(4)\mathbf{W}_4^H\mathbf{D}\mathbf{x}_{2,\mathrm{e}}$, where $\mathbf{D} = \mathrm{diag}(1, e^{j\frac{\pi}{8}}, e^{j\frac{2\pi}{8}}, e^{j\frac{3\pi}{8}})$.

IV. PEP Analysis for the Proposed Design

In this section, we will show that the PEP of our designed distributed concatenated recursive Alamouti-circulant STBC scheme proposed in Section III can achieve the lower bound given by Theorem 1.

A. Derivation of the PEP Formulations

By using (25) and taking the average of $P\left(\mathbf{s}_2 \to \mathbf{s}_2'|\mathbf{h}_1, \mathbf{h}_2\right)$ over the random vector $\mathbf{h}_2$, we have

$$P\left(\mathbf{s}_2 \to \mathbf{s}'_2|\mathbf{h}_1\right) = \frac{1}{\pi}\int_0^{\pi/2} \frac{d\theta}{\det\left(\mathbf{I}_{4N} + \frac{2P_2\Omega_1\beta}{4\sin^2\theta}\mathbf{K}_{2N}^H\overline{\mathbf{F}}_1^H\tilde{\mathbf{\Lambda}}^H\Sigma_1^{-1}\tilde{\mathbf{\Lambda}}\overline{\mathbf{F}}_1\mathbf{K}_{2N}\right)^{\frac{1}{2}}}, \tag{29}$$

where $\tilde{\mathbf{\Lambda}} = \mathrm{diag}(\Delta\mathcal{S}_2, \Delta\mathcal{S}_2^*)$, the difference matrix $\Delta\mathcal{S}_2 = \mathcal{S}_2 - \mathcal{S}_2'$. According to (24), the noise covariance matrix $\Sigma_1 = (2\beta\|\mathbf{h}_1\|^2 + 1)\sigma^2\mathbf{I}_4$, which yields

$$P\left(\mathbf{s}_2 \to \mathbf{s}'_2|\mathbf{h}_1\right) = \frac{1}{\pi}\int_0^{\pi/2} \frac{d\theta}{\det\left(\mathbf{I}_{4N} + \frac{2\beta_2(\theta)\rho}{2\beta\|\mathbf{h}_1\|^2+1}\overline{\mathbf{F}}_1^H\tilde{\mathbf{\Lambda}}^H\tilde{\mathbf{\Lambda}}\overline{\mathbf{F}}_1\right)^{\frac{1}{2}}} \tag{30}$$

where $\beta_2(\theta) = \left(4\sin^2\theta\right)^{-1}P_2\Omega_2\beta$ and SNR is defined as $\rho = \frac{1}{\sigma^2}$. As shown in Lemma 1, (30) is equivalent to

$$P\left(\mathbf{s}_2 \to \mathbf{s}_2'|\mathbf{h}_1\right) = \frac{1}{\pi}\int_0^{\pi/2} \frac{d\theta}{\det\left(\mathbf{I}_{2N} + \frac{2\beta_2(\theta)\rho}{2P_r\|\mathbf{h}_1\|^2+1}\mathbf{F}_1\mathbf{F}_1^H\Delta\mathcal{S}_2^H\Delta\mathcal{S}_2\right)}. \tag{31}$$

By taking the average of (31) over $\mathbf{h}_1$, we attain the following lemma, which will help to give our final Theorem 2.

Lemma 2: For a $2N \times 2N$ semi-definite matrix $\mathbf{Q}$ with full rank, if we let $J(\mathbf{h}_1, \mathbf{Q}) = \det(\mathbf{I}_{4N} + \alpha(\theta, \rho)\Lambda\mathbf{Q}\Lambda^H)^{-1}$ and $F(\rho, \mathbf{Q}) = \mathbb{E}_{\mathbf{h}_1}[J(\mathbf{h}_1, \mathbf{Q})]$, then $F(\rho, \mathbf{Q})$ has the following asymptotic formula when SNR is large,

$$F(\rho, \mathbf{Q}) = \frac{\ln^N\rho}{\det(\mathbf{I}_{2N} + \Omega_1\alpha(\theta, \rho)\mathbf{Q})} + O\left(\frac{\ln^{N-1}\rho}{\rho^{2N}}\right).$$

where $\Lambda = \mathrm{diag}(\mathbf{h}_{1,1}, \mathbf{h}_{1,1}, \mathbf{h}_{1,2}, \mathbf{h}_{1,2}, \cdots, \mathbf{h}_{1,N}, \mathbf{h}_{1,N})$ is a $4N \times 2N$ channel matrix, $N = 2^b$ and b is a nonnegative integer. We assume that all $h_{1,l}$, $l = 1, \cdots, 2N$, are *i.i.d.* complex Gaussian random variables, each of which has zero mean and variance Ω_1. In addition, $\alpha(\theta, \rho) = O(\rho)$ with ρ being the SNR. ■

Proof: See Appendix C.

Theorem 2: By using the proposed distributed concatenated recursive Alamouti-circulant STBC scheme, the PEP at $T_{\tilde{k}}$ for the TWRN with $N = 2^b$ relays with each having two antennas is given by

$$P\left(\mathbf{s}_k \to \mathbf{s}_k'\right) = \frac{2^{2N-1}(4N-1)!!}{(4N)!!N^{2N}P_k^{2N}\Omega_1^{2N}\Omega_2^{2N}\beta^{2N}\det(\Delta\mathcal{S}_k^H\Delta\mathcal{S}_k)}\frac{\ln^N\rho}{\rho^{2N}} + O\left(\frac{\ln^{N-1}\rho}{\rho^{2N}}\right). \tag{32}$$

■

Proof: See Appendix D.

We would like to make the following observations on Theorem 2. Firstly, the code design presented in this paper enables the ML detector of the receiver to achieve the diversity-gain lower bound given in Theorem 1. Secondly, in addition to the maximum diversity gain, the asymptotic PEP performance is dominated by $\min\det(\Delta\mathcal{S}_k^H\Delta\mathcal{S}_k)$, which, following the concept from the MIMO system, is called *coding gain*. Furthermore, to check the correctness of the PEP performance presented in *Theorem* 2, we perform the numerical simulations and compare them with theoretical results as presented in Section V.

B. Design of the Transmission Signal for Achieving the Minimum PEP

The PEP leads to two design criteria for space-time codes [47]: *Full Rank Criterion* (maximize the minimum rank of the difference matrix over all pairs of distinct codewords) and *Product Criterion* (maximize the minimum value of the determinant over all pairs of distinct codewords). Since $\det(\Delta\mathcal{S}_2^H\Delta\mathcal{S}_2) = \det^2(\Delta\mathcal{S}_2)$ and $\Delta\mathcal{S}_2 = \mathrm{diag}(\Delta\mathcal{S}_{2,1}, \Delta\mathcal{S}_{2,2}, \cdots, \Delta\mathcal{S}_{2,N})$, $\Delta\mathcal{S}_{2,j} = \begin{bmatrix}\Delta\chi_{k,2j-1} & -\Delta\chi_{k,2j}\\ \Delta\chi_{k,2j}^* & \Delta\chi_{k,2j-1}^*\end{bmatrix}$, we define the cost function as

$$\lambda(\mathbf{s}_2, \mathbf{s}_2') = \det(\Delta\mathcal{S}_2) = \prod_{j=1}^{N}\det(\Delta\mathcal{S}_{2,j}) = \prod_{j=1}^{N}\left(|\Delta\chi_{2,2j-1}|^2 + |\Delta\chi_{2,2j}|^2\right). \tag{33}$$

From (33), we can see that the determinant can be zero if $N > 1$ and $\mathbf{s}_k$ is chosen directly from the M-QAM constellation sets. To solve this problem, we properly choose the rotation angle ϕ_i of the $2N$ transmitted signals and the precoding matrix $\mathbf{W}$ to ensure that the worst case of the determinant is maximized. Thus, the solution to this optimization problem can be obtained as following.

$$d_2 = \max_{\phi_i, i=1,\cdots,N-1}\left(\min_{\mathbf{x}_2\neq\mathbf{x}_2'}\lambda(\phi_i, \mathbf{x}_2, \mathbf{x}_2')\right) = \max_{\phi_i, i=1,\cdots,N-1}\left(\min_{\mathbf{x}_2\neq\mathbf{x}_2'}\det(\Delta\mathcal{S}_2)\right). \tag{34}$$

Due to $\boldsymbol{\chi}_2 = \mathbf{T}_b\mathbf{s}_2 = \mathbf{T}_b\mathbf{E}\mathrm{diag}(\mathbf{P}, \mathbf{P})\mathbf{E}^H\mathbf{x}_2$, which is divided into $\boldsymbol{\chi}_{2,\mathrm{o}} = \frac{1}{\sqrt{N}}\mathrm{Hadamard}(N)\mathbf{s}_{2,\mathrm{o}} = \mathbf{W}_N^H\mathbf{D}\mathbf{x}_{2,\mathrm{o}}$ and $\boldsymbol{\chi}_{2,\mathrm{e}} = \mathbf{W}_N^H\mathbf{D}\mathbf{x}_{2,\mathrm{e}}$, we can see that

1) When $N = 1$, $\lambda(\mathbf{s}_2, \mathbf{s}_2') = \lambda(\mathbf{x}_2, \mathbf{x}_2') = |\Delta x_{2,1}|^2 + |\Delta x_{2,2}|^2$. It can be obtained that $d_2 = d_{min}^2$, where d_{min} denotes the minimum Euclidean distance of M-QAM constellation $\mathcal{Q}$ with unit power.
2) When $N = 2$, the determinant is simplified to

$$\max_{\phi_1}\left(\min_{\mathbf{x}_2\neq\mathbf{x}_2'}\lambda(\phi_1, \mathbf{x}_2, \mathbf{x}_2')\right) = \frac{1}{4}\max_{\phi_1}\min_{\mathbf{x}_2\neq\mathbf{x}_2'}\left((|\Delta x_{2,1} + e^{j\phi_1}\Delta x_{2,3}|^2 + |\Delta x_{2,2} + e^{j\phi_1}\Delta x_{2,4}|^2)\cdot(|\Delta x_{2,1} - e^{j\phi_1}\Delta x_{2,3}|^2 + |\Delta x_{2,2} - e^{j\phi_1}\Delta x_{2,4}|^2)\right). \tag{35}$$

It turns to be the same optimization problem as the conventional STBC signal design in [48]. It can be proved that, for M-QAM, $\phi_1 = \pi/4$ is the optimal angle to maximize the worst case of the determinant.

3) When $N = 2^b$, $b \geq 2$, as analyzed in [49], the transmitted signal $\mathbf{s}_2$ should be precoded by $\mathbf{s}_{2,\mathrm{o}} = \frac{1}{\sqrt{N}}\mathrm{Hadamard}(N)\mathbf{W}_N^H\mathbf{D}\mathbf{x}_{2,\mathrm{o}}$ and $\mathbf{s}_{2,\mathrm{e}} = \frac{1}{\sqrt{N}}\mathrm{Hadamard}(N)\mathbf{W}_N^H\mathbf{D}\mathbf{x}_{2,\mathrm{e}}$, where $x_l \in \mathcal{Q}$ is the constellation symbol vector. $\mathbf{D} = \mathrm{diag}(1, e^{j\phi_1}, \cdots, e^{j(N-1)\phi_1})$ and $\mathbf{W}_N$ is a discrete Fourier matrix, i.e., $\mathbf{W}_N(p,q) = \frac{1}{\sqrt{N}}e^{-j2\pi pq/N}$, $p, q = 1, \cdots, N$. To ensure the full rank of the difference matrix and maximize its determinant, $N-1$ rotation angles are required, i.e., $\phi_i = i \times \phi_1, i = 2, \cdots, N-1$. Thus, the solution to this optimization problem becomes

$$\max_{\phi_i}\Big(\min_{\mathbf{x}_2 \neq \mathbf{x}_2'} \lambda(\phi_1, \mathbf{x}_2, \mathbf{x}_2')\Big) = \max_{\phi_i, i=1,\cdots,N-1} \min_{\mathbf{x}_2 \neq \mathbf{x}_2'} \prod_{p=1}^{N} \left(|\mathbf{w}_p\mathbf{D}\Delta\mathbf{x}_{2,\mathrm{e}}|^2 + |\mathbf{w}_p\mathbf{D}\Delta\mathbf{x}_{2,\mathrm{o}}|^2\right), \tag{36}$$

where $\mathbf{w}_p$ denotes the p-th row of the matrix $\mathbf{W}_N^H$. For M-QAM, the optimal rotation angle is determined as $\phi_1 = \frac{\pi}{2N}$. The detailed proof can refer to the *Corollary* 1 and [49, Th. 5].

V. Simulations

Throughout the simulations, we assume that both of the two source nodes know perfect CSI, whereas the relay nodes only know the first and second order statistics values of the source-relay channels. Firstly, we carry our computer simulations and examine error performance by comparing the following two half-duplex TWRNs, which have almost the same complexity at relays:

(a) The TWRN composed of two source nodes and $2N$ relay nodes with each employing one antenna [15]–[17], [29], where the transmitted signal at the i-th relay is also designed to be a linear function of its received signal and its conjugate. For the networks with 2 relays and 4 relays, the corresponding DSTBCs have been presented in [29].

(b) The TWRN assisted by N dual-antenna relays using the code design proposed in Section III, where the optimized angle rotation and precoding are adopted to ensure the expected performance.

Fig. 3 gives the bit error rate (BER) comparison of network (a) and network (b) by using optimal power allocation (OPA) over the symmetric channels with $\Omega_1 = \Omega_2 = 1$, where the BER curves are obtained by averaging the BER values at two source nodes. It can be observed from Fig. 3 that network (b) outperforms network (a) in the whole SNR region considered when the two networks have the same antenna number. The superiority of our network (b) over network (a) becomes more obvious when SNR is increasing, owing to the fact that the slopes for network (b) always outperform those for network (a). This observation is consistent with our asymptotic PEP analysis, i.e., the PEP function for the network (a) is proportional to $\rho^{-2N}\ln^{2N}\rho$, whereas that for the network (b) is proportional to $\rho^{-2N}\ln^{N}\rho$ when $2N$ relay antennas are used. In addition, we can also observe that the BER performance of OPA outperforms that of conventional equal power allocation (EPA).

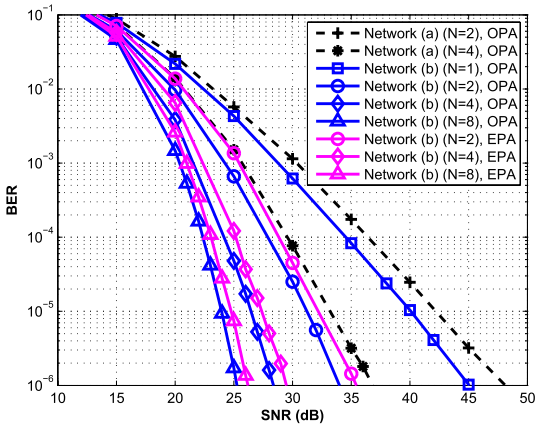

Fig. 3. Average BER performance comparison of network (a) and network (b) over symmetric channels $\Omega_1 = \Omega_2 = 1$, where $N = 1, 2, 4, 8$ and 4-QAM are considered. OPA: $P_1 = P_2 = \frac{1}{4}, P_r = \frac{1}{4N}$; EPA: $P_1 = P_2 = \frac{1}{3}, P_r = \frac{1}{6N}$.

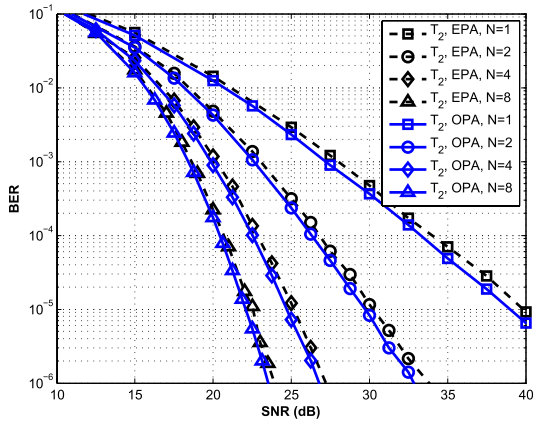

Fig. 4. BER of $\mathrm{T}_1 \rightarrow \mathrm{T}_2$, where $N = 1, 2, 4, 8$, $\Omega_1 = 1, \Omega_2 = 3$ and 4-QAM. OPA: $P_1 = \frac{3-\sqrt{3}}{4}$, $P_2 = \frac{\sqrt{3}-1}{4}$ and $P_r = \frac{1}{4N}$, EPA: $P_1 = P_2 = \frac{1}{3}$ and $P_r = \frac{1}{6N}$.

Secondly, we further illustrate the impact of power allocation on the BER performance of the proposed network (b) over asymmetric channels. Fig. 4 and Fig. 5 demonstrate the BER at T_2 and T_1 respectively under a typical asymmetric channel, i.e., $\Omega_1 = 1$ and $\Omega_2 = 3$. We can observe from these two figures that the OPA given by Proposition 2 indeed enhances the error performance of the whole network, compared with conventional equal power allocation (EPA). Specifically, at the BER of 10^{-4}, the SNR gains of OPA over EPA are about 0.3 dB$-$0.8 dB for $\mathrm{T}_1 \rightarrow \mathrm{T}_2$ link, whilst 1.5 dB$-$2.5 dB for the reverse $\mathrm{T}_2 \rightarrow \mathrm{T}_1$ link. Power loading is more important for the terminal having poor channel condition with relay nodes. In addition, it can be concluded that power loading doesn't affect the achieved diversity gain due to the fact that all the BER curves with the same relay number are parallel in high SNR region.

Thirdly, from the view of the pure STBC design, we can see that our proposed concatenated recursive Alamouti-circulant

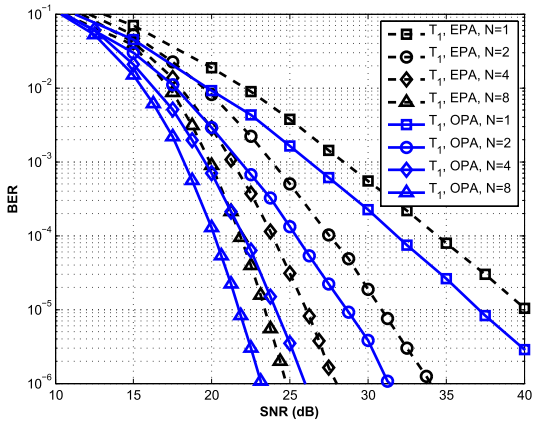

Fig. 5. BER of $T_2 \rightarrow T_1$, the same conditions as Fig. 4 are adopted.

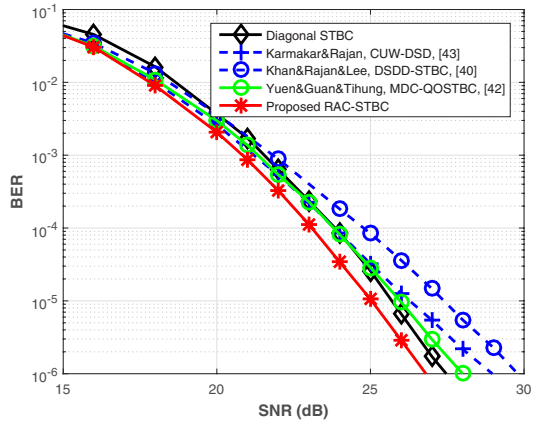

Fig. 6. BER performance comparison of different STBCs for MISO system with eight transmit antennas, 4-QAM.

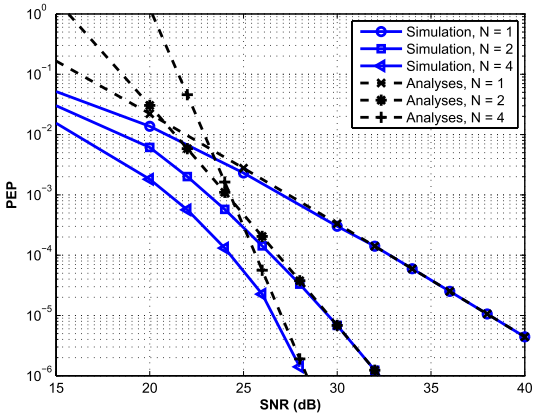

Fig. 7. Comparisons of the simulated PEP and the theoretical PEP given in *Theorem* 2 over channel $\Omega_1 = \Omega_2 = 1$, where OPA and 4-QAM are considered.

STBC (RAC-STBC) can also used for MISO systems. In this part, we carry the computer simulations and examine error performance compared with the existed four rate-one STBCs [40], [42], [43], where MISO system with eight transmit antennas is considered. From Fig. 6, we can observe that in this case, our proposed RAC-STBC scheme outperforms the other STBCs in the considered SNR region, e.g., about 1dB performance improvement at least at BER 10^{-4}. Furthermore, our design is a general rate-one design for any transmit antenna number with a power of two, i.e., it can achieve the full diversity and the maximum coding gain without computer searching.

Finally, we perform the numerical PEP simulations under the assumption of $\mathbf{s}'_k$ being the nearest one of the transmitted vector $\mathbf{s}_k$ with only one different adjacent constellation element. From the simulated PEP given in Fig.7, we can see that the theoretical derivation in *Theorem* 2 can accurately reflect the PEP performance when SNR is large.

VI. Conclusion

In this paper, we have first derived a tight diversity-gain lower bound with the ML receiver for the general distributed linear dispersion coded TWRN consisting of the two sources with each equipped with a single antenna and N relays with each employing two antennas. This lower bound reveals that the PEP cannot decay faster than $\ln^N \mathrm{SNR}/\mathrm{SNR}^{-2N}$ when SNR is high. Then, by carefully making use of the Alamouti coding scheme to combine the received signals at the relay nodes, we have presented an optimal DSTBC design for such networks with $N = 2^b$ relays, resulting in the fact that the equivalent channel at each source node is the RAC matrix, of which each block being the product of the two Alamouti channel matrices, and the equivalent noise at each source node is white Gaussian. Asymptotic PEP formulae have been obtained to show that the maximum diversity gain function, i.e., the diversity-gain lower bound, $\ln^N \mathrm{SNR}/\mathrm{SNR}^{2N}$, as well as the maximum coding gain are achieved by using our design. In addition, to further improve error performance, we have attained the optimal power allocation maximizing the worst-link received SNR. Comprehensive computer simulations have demonstrated that the code presented in this paper, along with the optimal power loading, provides substantial error performance gains over the currently-available DSTBCs in literature. Overall, because of the product of the two Alamouti coding matrices in the equivalent channel matrices and because of all those aforementioned optimality, we have called such a code *optimal distributed concatenated recursive Alamouti-circulant STBC*.

Appendix

A. Some Necessary Properties

In order to prove our results, we first recall some basic properties on matrix theory [46] which will be required in our derivation.

Proposition 3: Sylvester's determinant theorem states that for $\mathbf{A}$, an m-by-n matrix, and $\mathbf{B}$, an n-by-m matrix [46]:

$$\det(\mathbf{I}_m + \mathbf{AB}) = \det(\mathbf{I}_n + \mathbf{BA}).$$

Proposition 4: For any two $N \times N$ positive definite matrices $\mathbf{A}$ and $\mathbf{B}$, if $\mathbf{A} \succeq \mathbf{B}$, then,

$$\det(\mathbf{A}) \geq \det(\mathbf{B}).$$

Proposition 5: Let $\mathbf{K} = \begin{pmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{pmatrix}$ be a positive semi-definite matrix. Then, the following two statements are true.

1) If $\mathbf{K}_{11}$ is invertible, then, its Schur complementary matrix $\mathbf{K}_{22} - \mathbf{K}_{21}\mathbf{K}_{11}^{-1}\mathbf{K}_{12}$ is also a positive semi-definite matrix and $\det(\mathbf{K}) = \det(\mathbf{K}_{11}) \det(\mathbf{K}_{22} - \mathbf{K}_{21}\mathbf{K}_{11}^{-1}\mathbf{K}_{12})$.

2) If $\mathbf{K}_{22}$ is invertible, then, its Schur complementary matrix $\mathbf{K}_{11}-\mathbf{K}_{12}\mathbf{K}_{22}^{-1}\mathbf{K}_{21}$ is also a positive semi-definite matrix and $\det(\mathbf{K}) = \det(\mathbf{K}_{22})\det(\mathbf{K}_{11}-\mathbf{K}_{12}\mathbf{K}_{22}^{-1}\mathbf{K}_{21})$.

B. Proof of Proposition 1

1) When $b = 0$, i.e., $N = 1$, we directly have $\mathbf{T}_0 = \mathbf{T}_{0,1} = \mathbf{I}_2$ and $\mathbf{g}_0 = \mathbf{G}_0$, thus $\mathbf{D}_0 = \mathbf{T}_0^H\mathbf{D}_0\mathbf{T}_0 = \mathrm{diag}(\mathbf{T}_{0,1}\mathbf{g}_0) = \mathbf{G}_0$.

2) Now, we assume that $\mathbf{G}_n = \mathbf{T}_n^H\mathbf{D}_n\mathbf{T}_n$ holds for any RACM $\mathbf{G}_n \in \mathrm{RACM}_{n+1}$. Then, for $b = n+1$, we prove that a RACM $\mathbf{G}_{n+1} \in \mathrm{RACM}_{n+2}$ can be expressed as $\mathbf{G}_{n+1} = \mathbf{T}_{n+1}^H\mathbf{D}_{n+1}\mathbf{T}_{n+1}$. To prove this equation, we firstly recall that $\mathbf{G}_{n+1} = \begin{bmatrix}\mathbf{G}_n^1 & \mathbf{G}_n^2\\ \mathbf{G}_n^2 & \mathbf{G}_n^1\end{bmatrix}$, where $\mathbf{G}_n^\ell \in \mathrm{RACM}_{n+1}$ for $\ell = 1, 2$. According to our assumption, $\mathbf{G}_n^\ell = \mathbf{T}_n^H\mathbf{D}_n^\ell\mathbf{T}_n$, where $\mathbf{D}_n^\ell = \sqrt{2^n}\mathrm{diag}(\mathbf{T}_{n,1}\mathbf{g}_n^\ell, \mathbf{T}_{n,2}\mathbf{g}_n^\ell, \cdots, \mathbf{T}_{n,2^n}\mathbf{g}_n^\ell)$, in which $\mathbf{g}_n^\ell$ is the first two columns of $\mathbf{G}_n^\ell$. Since $\mathbf{T}_{n+1}$ can be written as $\mathbf{T}_{n+1} = \frac{1}{\sqrt{2}}\begin{bmatrix}\mathbf{T}_n & \mathbf{T}_n\\ \mathbf{T}_n & -\mathbf{T}_n\end{bmatrix}$, we readily have

$$\begin{aligned}\mathbf{D}_{n+1} &= \sqrt{2^{n+1}}\mathrm{diag}(\mathbf{T}_{n+1,1}\mathbf{g}_{n+1}, \mathbf{T}_{n+1,2}\mathbf{g}_{n+1}, \cdots, \\ &\qquad \mathbf{T}_{n+1,2^{n+1}}\mathbf{g}_{n+1})\\ &= \begin{bmatrix}\mathbf{D}_n^1 + \mathbf{D}_n^2 & \\ & \mathbf{D}_n^1 - \mathbf{D}_n^2\end{bmatrix}\end{aligned}$$

due to $\mathbf{g}_{n+1} = \left[\mathbf{g}_n^1;\ \mathbf{g}_n^2\right]$ and $\mathbf{T}_{n+1,j} = \begin{cases}\left[\mathbf{T}_{n,j}\ \ \mathbf{T}_{n,j}\right], & \text{if } j \le 2^n,\\ \left[\mathbf{T}_{n,j-2^n}\ \ -\mathbf{T}_{n,j-2^n}\right], & \text{else if } j > 2^n.\end{cases}$ Now, $\mathbf{G}_{n+1}$ can be rewritten as

$$\begin{aligned}\mathbf{G}_{n+1} &= \begin{bmatrix}\mathbf{T}_n^H\mathbf{D}_n^1\mathbf{T}_n & \mathbf{T}_n^H\mathbf{D}_n^2\mathbf{T}_n\\ \mathbf{T}_n^H\mathbf{D}_n^2\mathbf{T}_n & \mathbf{T}_n^H\mathbf{D}_n^1\mathbf{T}_n\end{bmatrix}\\ &= \frac{1}{2}\begin{bmatrix}\mathbf{T}_n^H & \mathbf{T}_n^H\\ \mathbf{T}_n^H & -\mathbf{T}_n^H\end{bmatrix}\begin{bmatrix}\mathbf{D}_n^1 + \mathbf{D}_n^2 & \\ & \mathbf{D}_n^1 - \mathbf{D}_n^2\end{bmatrix}\\ &\quad\cdot\begin{bmatrix}\mathbf{T}_n & \mathbf{T}_n\\ \mathbf{T}_n & -\mathbf{T}_n\end{bmatrix}\\ &= \mathbf{T}_{n+1}^H\mathbf{D}_{n+1}\mathbf{T}_{n+1}.\end{aligned}$$

This completes the proof of Proposition 1. □

C. Proof of Lemma 2

In the following, we let $t_j = \frac{1}{\Omega_1}(|h_{1,2j-1}|^2 + |h_{1,2j}|^2)$, i.e., the probabilistic density function of t_j is $t_j e^{-t_j}$. We prove this lemma by the induction on b.

1) When $b = 0$, i.e., $N = 1$, $\mathbf{Q} = q_{11}\mathbf{I}_2$, $\mathbf{D} = \|\mathbf{h}_{1,1}\|^2\mathbf{I}_2$ and hence, $F(\rho, \mathbf{Q})$ becomes

$$\begin{aligned}F_0(\rho, \mathbf{Q}) &= \int_0^\infty \frac{t_1 e^{-t_1}}{(1+\Omega_1\alpha(\theta,\rho)q_{11}t_1)^2}dt_1\\ &= \frac{\ln\rho + O(1)}{\Omega_1^2\alpha^2(\theta,\rho)q_{11}^2}\\ &= \frac{\ln\rho}{\det(\mathbf{I}_2 + \Omega_1\alpha(\theta,\rho)\mathbf{Q})} + O\left(\frac{1}{\rho^2}\right),\end{aligned}$$

where we use $\frac{1}{\rho} = \frac{1}{1+\rho} + O(\frac{1}{\rho^2})$.

2) Assume that when $b = n$, i.e., $N = 2^n$ relays, we have

$$F_n(\rho, \mathbf{Q}) = \frac{\ln^N\rho}{\det(\mathbf{I}_{2N} + \Omega_1\alpha(\theta,\rho)\mathbf{Q})} + O\left(\frac{\ln^{N-1}\rho}{\rho^{2N}}\right). \quad (37)$$

3) When $b = n+1$, i.e., $2N$ relays, we prove

$$\begin{aligned}&F_{n+1}(\rho, \mathbf{Q})\\ &\quad= \frac{\ln^{2N}\rho}{\det(\mathbf{I}_{4N} + \Omega_1\alpha(\theta,\rho)\mathbf{Q})} + O\left(\frac{\ln^{2N-1}\rho}{\rho^{4N}}\right). \quad (38)\end{aligned}$$

By Proposition 5

$$\begin{aligned}J_{n+1}(\mathbf{h}_1, \mathbf{Q}) &= \det^{-1}(\mathbf{I}_{8N} + \alpha(\theta,\rho)\Lambda\mathbf{Q}\Lambda^H)\\ &= \det^{-1}\left(\mathbf{I}_{8N} + \alpha(\theta,\rho)\begin{bmatrix}\Lambda_1\mathbf{Q}_{11}\Lambda_1^H & \Lambda_1\mathbf{Q}_{12}\Lambda_2^H\\ \Lambda_2\mathbf{Q}_{21}\Lambda_1^H & \Lambda_2\mathbf{Q}_{22}\Lambda_2^H\end{bmatrix}\right) \quad (39)\\ &= \det^{-1}(\mathbf{I}_{4N} + \alpha(\theta,\rho)\Lambda_2\mathbf{Q}_{22}\Lambda_2^H)\\ &\qquad\cdot\det^{-1}(\mathbf{I}_{4N} + \alpha(\theta,\rho)\Lambda_1\bar{\mathbf{Q}}_{22}\Lambda_1^H), \quad (40)\end{aligned}$$

where $\mathbf{Q}_{11} = \mathbf{Q}_{22}$, $\mathbf{Q}_{12} = \mathbf{Q}_{21}$, and $\bar{\mathbf{Q}}_{22} = \mathbf{Q}_{11} - \alpha(\theta,\rho)\mathbf{Q}_{12}\Lambda_2^H(\mathbf{I}_{4N} + \alpha(\theta,\rho)\Lambda_2\mathbf{Q}_{22}\Lambda_2^H)^{-1}\Lambda_2\mathbf{Q}_{21}$. We can use the assumption $F_n(\rho, \mathbf{Q})$ to obtain that

$$\begin{aligned}F_{n+1}(\rho, \mathbf{Q}) &= \mathbb{E}_{\mathbf{h}_1}[J_{n+1}(\mathbf{h}_1, \mathbf{Q})]\\ &= \mathbb{E}_{\Lambda_2}[\det^{-1}(\mathbf{I}_{4N} + \alpha(\theta,\rho)\Lambda_2\mathbf{Q}_{22}\Lambda_2^H)\\ &\qquad\cdot\mathbb{E}_{\Lambda_1}\left[\det^{-1}(\mathbf{I}_{4N} + \alpha(\theta,\rho)\Lambda_1\bar{\mathbf{Q}}_{22}\Lambda_1^H)\right]\\ &= \mathbb{E}_{\Lambda_2}\left[\frac{1}{\det(\mathbf{I}_{4N} + \alpha(\theta,\rho)\Lambda_2\mathbf{Q}_{22}\Lambda_2^H)}\right.\\ &\qquad\left.\cdot\left(\frac{\ln^N\rho}{\det(\mathbf{I}_{2N} + \Omega_1\alpha(\theta,\rho)\overline{\mathbf{Q}}_{22})} + O\left(\frac{\ln^{N-1}\rho}{\rho^{2N}}\right)\right)\right]\\ &= \mathbb{E}_{\Lambda_2}\left[\frac{1}{\det(\mathbf{I}_{4N} + \alpha(\theta,\rho)\Lambda_2\mathbf{Q}_{22}\Lambda_2^H)}\right.\\ &\qquad\left.\cdot\frac{\ln^N\rho}{\det(\mathbf{I}_{4N} + \Omega_1\alpha(\theta,\rho)\mathbf{C}\overline{\mathbf{Q}}_{22}\mathbf{C}^H)}\right] + O\left(\frac{\ln^{2N-1}\rho}{\rho^{4N}}\right),\end{aligned}$$

where $\mathbf{C} = \frac{1}{\sqrt{2}}\mathbf{I}_{2N}\otimes\mathbf{e}_2$ and $\mathbf{e}_2 = [1, 1]^T$. According to Proposition 5, $F_{n+1}(\rho, \mathbf{Q})$ arrives at

$$\begin{aligned}F_{n+1}(\rho, \mathbf{Q}) &= \mathbb{E}_{\Lambda_2}\left[\frac{\ln^N\rho}{\det(\mathbf{I}_{8N} + \Omega_1\alpha(\theta,\rho)\bar{\Lambda}\mathbf{Q}\bar{\Lambda}^H)}\right]\\ &\qquad + O\left(\frac{\ln^{2N-1}\rho}{\rho^{4N}}\right), \quad (41)\end{aligned}$$

where $\bar{\Lambda} = \mathrm{diag}(\sqrt{\Omega_1}\mathbf{C}, \Lambda_2)$. Again, we have

$$\begin{aligned}F_{n+1}(\rho, \mathbf{Q}) &= \mathbb{E}_{\Lambda_2}\left[\frac{\ln^N\rho}{\det(\mathbf{I}_{4N} + \Omega_1\alpha(\theta,\rho)\mathbf{C}\mathbf{Q}_{11}\mathbf{C}^H)}\right.\\ &\qquad\left.\cdot\frac{1}{\det(\mathbf{I}_{4N} + \alpha(\theta,\rho)\Lambda_2\overline{\mathbf{Q}}_{11}\Lambda_2^H)}\right]\\ &\qquad + O\left(\frac{\ln^{2N-1}\rho}{\rho^{4N}}\right)\end{aligned}$$

where $\bar{\mathbf{Q}}_{11} = \mathbf{Q}_{22} - \Omega_1\alpha(\theta,\rho)\mathbf{Q}_{21}\mathbf{C}^H(\mathbf{I}_{4N} + \Omega_1\alpha(\theta,\rho)\mathbf{C}\mathbf{Q}_{11}\mathbf{C}^H)^{-1}\mathbf{C}\mathbf{Q}_{12}$. By using the assumption (37)

again, we have

$$F_{n+1}(\rho,\mathbf{Q}) = \frac{\ln^{2N}\rho}{\det(\mathbf{I}_{8N}+\boldsymbol{\Omega}_1\alpha(\theta,\rho)\mathrm{diag}(\mathbf{C},\mathbf{C})\mathbf{Q}\mathrm{diag}(\mathbf{C}^H,\mathbf{C}^H))} + O\left(\frac{\ln^{2N-1}\rho}{\rho^{4N}}\right),$$

which is equivalent with

$$F_{n+1}(\rho,\mathbf{Q}) = \frac{\ln^{2N}\rho}{\det(\mathbf{I}_{4N}+\boldsymbol{\Omega}_1\alpha(\theta,\rho)\mathbf{Q})} + O\left(\frac{\ln^{2N-1}\rho}{\rho^{4N}}\right). \tag{42}$$

This completes the proof of Lemma 1. □

D. Proof of Theorem 2

We prove Theorem 2 by deriving the lower bound and the upper bound of PEP respectively.

1) Lower Bound: In this case, the conditional PEP given by (31) satisfies

$$\begin{aligned} &P\left(\mathbf{s}_2\to\mathbf{s}_2'|\mathbf{h}_1\right)\\ &\geq \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\det\left(\mathbf{I}_{2N}+2\beta_2(\theta)\rho\mathbf{F}_1\mathbf{F}_1^H\Delta\mathcal{S}_2^H\Delta\mathcal{S}_2\right)}\\ &= \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\det(\mathbf{I}_{2N}+2\beta_2(\theta)\rho\underbrace{\mathbf{T}_b\mathbf{F}_1\mathbf{F}_1^H\mathbf{T}_b^H}_{\mathbf{D}}\underbrace{\mathbf{T}_b\Delta\mathcal{S}_2^H\Delta\mathcal{S}_2\mathbf{T}_b^H}_{\mathbf{Q}})}\\ &= P_{\mathrm{LB}}\left(\mathbf{s}_2\to\mathbf{s}_2'|\mathbf{h}_1\right), \end{aligned} \tag{43}$$

where $\Delta\mathcal{S}_2 = \mathrm{diag}(\Delta\mathcal{S}_{2,1},\ \Delta\mathcal{S}_{2,2},\ \cdots,\ \Delta\mathcal{S}_{2,N})$. Let $\Lambda = \mathrm{diag}(\mathbf{h}_{1,1},\ \mathbf{h}_{1,1},\ \mathbf{h}_{1,2},\ \mathbf{h}_{1,2},\ \cdots,\ \mathbf{h}_{1,N},\ \mathbf{h}_{1,N})$, $\mathbf{D} = \mathbf{T}_b\mathbf{F}_1\mathbf{F}_1^H\mathbf{T}_b^H = N\Lambda^H\Lambda = N\mathrm{diag}(\|\mathbf{h}_{1,1}\|^2\mathbf{I}_2,\ \|\mathbf{h}_{1,2}\|^2\mathbf{I}_2,\cdots,\|\mathbf{h}_{1,N}\|^2\mathbf{I}_2)$ and $\mathbf{Q} = \mathbf{T}_b\Delta\mathcal{S}_2^H\Delta\mathcal{S}_2\mathbf{T}_b^H$ are also semi-positive matrices. According to Proposition 3, (43) becomes

$$P_{\mathrm{LB}}\left(\mathbf{s}_2\to\mathbf{s}_2'|\mathbf{h}_1\right) = \frac{1}{\pi}\int_0^{\pi/2}\frac{d\theta}{\det\left(\mathbf{I}_{4N}+2N\beta_2(\theta)\rho\Lambda\mathbf{Q}\Lambda^H\right)}. \tag{44}$$

By using Lemma 2, we can write the lower bound as

$$\begin{aligned} P_{\mathrm{LB}}\left(\mathbf{s}_2\to\mathbf{s}_2'\right) &= \frac{1}{\pi}\int_0^{\pi/2}\mathbb{E}_{\mathbf{h}_1}\left[\frac{1}{\det\left(\mathbf{I}_{4N}+2N\beta_2(\theta)\rho\Lambda\mathbf{Q}\Lambda^H\right)}\right]d\theta\\ &= \frac{1}{\pi}\int_0^{\pi/2}\frac{\ln^N\rho}{\det\left(\mathbf{I}_{2N}+2N\boldsymbol{\Omega}_1\beta_2(\theta)\rho\mathbf{Q}\right)}d\theta + O\left(\frac{\ln^{N-1}\rho}{\rho^{2N}}\right). \end{aligned} \tag{45}$$

In addition, since we assume $\mathbf{Q}$ is a full rank matrix, we notice that $\det(\mathbf{I}_{2N}+2N\boldsymbol{\Omega}_1\beta_2(\theta)\rho\mathbf{Q}) = \prod_{\ell=1}^{2N}(1+2N\Omega_1\beta_2(\theta)\rho\lambda_\ell)$ and $\frac{1}{1+2N\Omega_1\beta_2(\theta)\rho\lambda_\ell} = \frac{1}{2N\Omega_1\beta_2(\theta)\rho\lambda_\ell} + O(\rho^{-2})$ with λ_ℓ is the ℓ-th non-zero eigenvalue of $\mathbf{Q}$, then we have

$$P_{\mathrm{LB}}\left(\mathbf{s}_2\to\mathbf{s}_2'\right) = \frac{1}{\pi}\int_0^{\pi/2}\frac{\ln^N\rho d\theta}{2^{2N}N^{2N}\boldsymbol{\Omega}_1^{2N}\beta_2^{2N}(\theta)\prod_{\ell=1}^{2N}\lambda_\ell\rho^{2N}} + O\left(\frac{\ln^{N-1}\rho}{\rho^{2N}}\right). \tag{46}$$

Replacing $\beta_2(\theta)$ with $\frac{P_2\Omega_2\beta}{4\sin^2\theta}$, we obtain

$$(46) = \frac{2^{2N-1}(4N-1)!!}{(4N)!!N^{2N}P_2^{2N}\boldsymbol{\Omega}_1^{2N}\boldsymbol{\Omega}_2^{2N}\beta^{2N}\prod_{\ell=1}^{2N}\lambda_\ell}\frac{\ln^N\rho}{\rho^{2N}} + O\left(\frac{\ln^{N-1}\rho}{\rho^{2N}}\right), \tag{47}$$

where we use $\int_0^{\pi/2}\sin^{4N}\theta d\theta = \frac{(4N-1)!!\pi}{(4N)!!2}$.

2) Upper Bound: To derive the upper bound of PEP, the conditional PEP given by (31) can be written as

$$\begin{aligned} &P\left(\mathbf{s}_2\to\mathbf{s}_2'|\mathbf{h}_1\right)\\ &= \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\frac{(2\beta\|\mathbf{h}_1\|^2+1)^{2N}d\theta}{\det\left((2\beta\|\mathbf{h}_1\|^2+1)\mathbf{I}_{2N}+2\beta_2(\theta)\rho\mathbf{F}\mathbf{F}^H\Delta\mathcal{S}_2^H\Delta\mathcal{S}_2\right)}\\ &\leq \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\frac{(2\beta\|\mathbf{h}_1\|^2+1)^{2N}d\theta}{\det\left(\mathbf{I}_{2N}+2\beta_2(\theta)\rho\mathbf{F}\mathbf{F}^H\Delta\mathcal{S}_2^H\Delta\mathcal{S}_2\right)}. \end{aligned} \tag{48}$$

Similarly, we define the upper bound of PEP as

$$\begin{aligned} &P_{\mathrm{UB}}\left(\mathbf{s}_2\to\mathbf{s}_2'\right)\\ &= \frac{1}{\pi}\int_0^{\pi/2}\mathbb{E}_{\mathbf{h}_1}\left[\frac{(2\beta\|\mathbf{h}_1\|^2+1)^{2N}}{\det\left(\mathbf{I}_{4N}+2N\beta_2(\theta)\rho\Lambda\mathbf{Q}\Lambda^H\right)}\right]d\theta. \end{aligned} \tag{49}$$

Notice that $(2\beta\|\mathbf{h}_1\|^2+1)^{2N}$ can be expanded into $(2\beta\|\mathbf{h}_1\|^2+1)^{2N} = 1 + f(\|\mathbf{h}_1\|^2) = 1 + \sum_{\ell=1}^{2N}C_{2N}^{\ell}2^{\ell}\beta^{\ell}\|\mathbf{h}_1\|^{2\ell}$, where $C_m^{\ell} = \frac{m!}{\ell!(m-\ell)!}$. Substituting this expansion into (49) yields

$$\begin{aligned} (49) &= \frac{1}{\pi}\int_0^{\pi/2}\mathbb{E}_{\mathbf{h}_1}\left[\frac{1}{\det\left(\mathbf{I}_{4N}+2N\beta_2(\theta)\rho\Lambda\mathbf{Q}\Lambda^H\right)}\right]d\theta\\ &\quad+\underbrace{\frac{1}{\pi}\int_0^{\pi/2}\mathbb{E}_{\mathbf{h}_1}\left[\frac{f(\|\mathbf{h}_1\|^2)}{\det\left(\mathbf{I}_{4N}+2N\beta_2(\theta)\rho\Lambda\mathbf{Q}\Lambda^H\right)}\right]d\theta}_{P_{1,h}(\rho)}. \end{aligned} \tag{50}$$

The first term of (50) is the same as (45), while the second term of (50) can be simplified as

$$\begin{aligned} P_{1,h}(\rho) &\leq \frac{1}{\pi}\int_0^{\pi/2}\mathbb{E}_{\mathbf{h}_1}\left[\frac{f(\|\mathbf{h}_1\|^2)}{\det\left(\mathbf{I}_{2N}+2N\beta_2(\theta)\rho\lambda_{\min}\Lambda^H\Lambda\right)}\right]d\theta\\ &= \frac{1}{\pi}\int_0^{\pi/2}\mathbb{E}_{\mathbf{h}_1}\left[\frac{\sum_{\ell=1}^{2N}C_{2N}^{\ell}2^{\ell}\beta^{\ell}\|\mathbf{h}_1\|^{2\ell}}{\prod_{j=1}^{N}\left(1+2N\beta_2(\theta)\rho\lambda_{\min}\|\mathbf{h}_{1,j}\|^2\right)^2}\right]d\theta. \end{aligned} \tag{51}$$

Due to $\|\mathbf{h}_1\|^{2\ell} = (\sum_{j=1}^{N}\|\mathbf{h}_{1,j}\|^2)^{\ell} = \sum_{\ell_1,\cdots,\ell_N}d_{\ell_1,\cdots,\ell_N}\prod_{j=1}^{N}\|\mathbf{h}_{1,j}\|^{2\ell_j}$, where $d_{\ell_1,\cdots,\ell_N}$ are constants which are independent of $\mathbf{h}_{1,j}$ for $j=1,\cdots,N$, ℓ_j are integers ranging from 0 to ℓ, we obtain

$$\begin{aligned} &\mathbb{E}_{\mathbf{h}_1}\left[\frac{\|\mathbf{h}_1\|^{2\ell}}{\prod_{j=1}^{N}\left(1+2N\beta_2(\theta)\rho\lambda_{\min}\|\mathbf{h}_{1,j}\|^2\right)^2}\right]\\ &\quad=\prod_{j=1}^{N}\mathbb{E}_{\mathbf{h}_{1,j}}\left[\frac{\|\mathbf{h}_{1,j}\|^{2\ell_j}}{\left(1+2N\beta_2(\theta)\rho\lambda_{\min}\|\mathbf{h}_{1,j}\|^2\right)^2}\right], \end{aligned} \tag{52}$$

which is obtained for that $\|\mathbf{h}_{1,j}\|^2$ are independent with each other. Due to

$$\mathbb{E}_{\mathbf{h}_{1,j}}\left[\frac{\|\mathbf{h}_{1,j}\|^{2\ell_j}}{\left(1+2N\beta_2(\theta)\rho\lambda_{\min}\|\mathbf{h}_{1,j}\|^2\right)^2}\right] = \begin{cases} O(\rho^{-2}), & \text{if } \ell_j > 0, \\ O(\rho^{-2}\ln\rho), & \text{else if } \ell_j = 0, \end{cases} \tag{53}$$

and at least one of ℓ_j is not zero, combining (52), (53) with (51) results in

$$P_{1,h}(\rho) = O(\rho^{-2N}\ln^{N-1}\rho). \tag{54}$$

Finally, substituting (54) and (47) into (50), we have

$$P_{\mathrm{UB}}\left(\mathbf{s}_2 \to \mathbf{s}_2'\right) = \frac{2^{2N-1}(4N-1)!!}{(4N)!!N^{2N}P_2^{2N}\Omega_1^{2N}\Omega_2^{2N}\beta^{2N}\prod_{\ell=1}^{2N}\lambda_\ell} \times\frac{\ln^N\rho}{\rho^{2N}} + O\left(\frac{\ln^{N-1}\rho}{\rho^{2N}}\right). \tag{55}$$

Combining (47) and (55) result in (32) with $k = 2$, which is the PEP at T_1. This completes the proof of Theorem 2. □

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REFERENCES

[1] Y. Yang, H. Hu, J. Xu, and G. Mao, "Relay technologies for WiMax and LTE-advanced mobile systems," *IEEE Commun. Mag.*, vol. 47, no. 10, pp. 100–105, Oct. 2009.

[2] Z. Sheng, K. K. Leung, and Z. Ding, "Cooperative wireless networks: From radio to network protocol designs," *IEEE Commun. Mag.*, vol. 49, no. 5, pp. 64–69, May 2011.

[3] A. Rankov and B. Wittneben, "Spectral efficient protocols for half-duplex fading relay channels," *IEEE J. Sel. Areas Commun.*, vol. 25, no. 2, pp. 379–389, Feb. 2007.

[4] M. K. Arti and M. R. Bhatnagar, "Two-way mobile satellite relaying: A beamforming and combining based approach," *IEEE Commun. Lett.*, vol. 18, no. 7, pp. 1187–1190, Jul. 2014.

[5] M. R. Bhatnagar, "Making two-way satellite relaying feasible: A differential modulation based approach," *IEEE Trans. Commun.*, vol. 63, no. 8, pp. 2836–2847, Aug. 2015.

[6] S. M. Alamouti, "A simple transmit diversity technique for wireless communications," *IEEE J. Sel. Areas Commun*, vol. 16, no. 8, pp. 1451–1458, Oct. 1998.

[7] T. P. Ren, Y. L. Guan, C. Yuen, E. Gunawan, and E. Y. Zhang, "Group-decodable space-time block codes with code rate > 1," *IEEE Trans. Commun.*, vol. 59, no. 4, pp. 987–997, Apr. 2011.

[8] T. P. Ren, Y. L. Guan, C. Yuen, and R. J. Shen, "Fast-group-decodable space-time block code," in *Proc. IEEE Inf. Theory Workshop Inf. Theory (ITW)*, Cairo, Egypt, Jan. 2010, pp. 1–5.

[9] T. P. Ren, Y. L. Guan, C. Yuen, and E. Y. Zhang, "Block-orthogonal space–time code structure and its impact on QRDM decoding complexity reduction," *IEEE J. Sel. Topics Signal Process.*, vol. 5, no. 8, pp. 1438–1450, Dec. 2011.

[10] J. N. Laneman, D. N. C. Tse, and G. W. Wornell, "Cooperative diversity in wireless networks: Efficient protocols and outage behavior," *IEEE Trans. Inf. Theory*, vol. 50, no. 12, pp. 3062–3080, Dec. 2004.

[11] Y. Jing and B. Hassibi, "Distributed space-time coding in wireless relay networks," *IEEE Trans. Wireless Commun.*, vol. 5, no. 12, pp. 3524–3536, Dec. 2006.

[12] T. Cui, F. Gao, T. Ho, and A. Nallanathan, "Distributed space–time coding for two-way wireless relay networks," *IEEE Trans. Signal Process.*, vol. 57, no. 2, pp. 658–671, Feb. 2009.

[13] Y. Han, S. H. Ting, C. K. Ho, and W. H. Chin, "High rate two-way amplify-and-forward half-duplex relaying with OSTBC," in *Proc. IEEE VTC Spring*, Singapore, May 2008, pp. 2426–2430.

[14] Y. Han, S. H. Ting, C. K. Ho, and W. H. Chin, "Performance bounds for two-way amplify-and-forward relaying," *IEEE Trans. Wireless Commun.*, vol. 8, no. 1, pp. 432–439, Jan. 2009.

[15] F. Abdurahman, A. Elazreg, and J. A. Chambers, "Distributed quasi-orthogonal space-time coding for two-way wireless relay networks," in *Proc. 7th Int. Symp. Wireless Commun. Syst. (ISWCS)*, York, U.K., Sep. 2010, pp. 413–416.

[16] U. N. Mannai, A. M. Elazreg, and J. A. Chambers, "Distributed closed-loop extended orthogonal space time block coding for two-way wireless relay networks," in *Proc. Int. Conf. Softw., Telecommun. Comput. Netw. (SoftCOM)*, Split, Croatia, Sep. 2010, pp. 190–194.

[17] M. A. Manna, F. Abdurahman, and J. Chambers, "Distributed quasi-orthogonal type space-time block coding with maximum distance property for two-way wireless relay networks," in *Proc. IEEE Int. Wireless Commun. Mobile Comput. Conf. (IWCMC)*, Jul. 2011, pp. 1704–1707.

[18] C. Yuen, W. H. Chin, Y. L. Guan, W. Chen, and T. Tee, "Bi-directional multi-antenna relay communications with wireless network coding," in *Proc. IEEE 69th Veh. Technol. Conf. (VTC Spring)*, May 2008, pp. 1385–1388.

[19] M. Eslamifar, W. H. Chin, C. Yuen, and Y. L. Guan, "Performance analysis of two-step bi-directional relaying with multiple antennas," *IEEE Trans. Wireless Commun.*, vol. 11, no. 12, pp. 4237–4242, Dec. 2012.

[20] R. H. Y. Louie, Y. Li, and B. Vucetic, "Practical physical layer network coding for two-way relay channels: Performance analysis and comparison," *IEEE Trans. Wireless Commun.*, vol. 9, no. 2, pp. 764–777, Feb. 2010.

[21] T. D. Trung, C. Yuen, H.-J. Zepernick, and X. Lei, "Average sum-rate of distributed Alamouti space-time scheme in two-way amplify-and-forward relay networks," in *Proc. IEEE GLOBECOM Workshops*, Miami, FL, USA, Dec. 2010, pp. 79–83.

[22] T. Q. Duong, H. Q. Ngo, H.-J. Zepernick, and A. Nallanathan, "Distributed space-time coding in two-way fixed gain relay networks over Nakagami-m fading," in *Proc. IEEE Int. Commun. Conf. (ICC)*, Ottawa, ON, Canada, Jun. 2012, pp. 3521–3525.

[23] S. J. Alabed, J. M. Paredes, and A. B. Gershman, "A simple distributed space-time coded strategy for two-way relay channels," *IEEE Trans. Wireless Commun.*, vol. 11, no. 4, pp. 1260–1265, Apr. 2012.

[24] F.-K. Gong, J.-K. Zhang, and J.-H. Ge, "Distributed concatenated Alamouti codes for two-way relaying networks," *IEEE Wireless Commun. Lett.*, vol. 1, no. 3, pp. 197–200, Jun. 2012.

[25] F.-K. Gong, J.-K. Zhang, and J.-H. Ge, "Novel distributed quasi-orthogonal space-time block codes for two-way two-antenna relay networks," *IEEE Trans. Wireless Commun.*, vol. 12, no. 9, pp. 4338–4349, Sep. 2013.

[26] M.-T. O. El Astal, B. P. Salmon, and J. C. Olivier, "Distributed space-time block coding for two-way wireless relaying networks: Improved performance under imperfect synchronization," in *Proc. IEEE Wireless Commun. Netw. Conf. (WCNC)*, vol. 1. Istanbul, Turkey, Apr. 2014, pp. 1176–1181.

[27] K. G. Unnikrishnan and B. S. Rajan, "Space–time coded spatial modulated physical layer network coding for two-way relaying," *IEEE Trans. Wireless Commun.*, vol. 14, no. 1, pp. 331–342, Jan. 2015.

[28] F.-K. Gong, J.-K. Zhang, and J.-H. Ge, "Optimal distributed concatenated space-time block codes for two-way relaying networks," in *Proc. IEEE Int. Conf. Acoust., Speech Signal Process. (ICASSP)*, vol. 1. Vancouver, BC, Canada, May 2013, pp. 4978–4982.

[29] Y. Jing and H. Jafarkhani, "Using orthogonal and quasi-orthogonal designs in wireless relay networks," *IEEE Trans. Inf. Theory*, vol. 53, no. 11, pp. 4106–4118, Nov. 2007.

[30] J. Harshan and B. S. Rajan, "Co-ordinate interleaved distributed space-time coding for two-antenna-relays networks," *IEEE Trans. Wireless Commun.*, vol. 8, no. 4, pp. 1783–1791, Apr. 2009.

[31] R. Zhang, C. C. Chai, and Y.-C. Liang, "Joint beamforming and power control for multiantenna relay broadcast channel with QoS constraints," *IEEE Trans. Signal Process.*, vol. 57, no. 2, pp. 726–737, Feb. 2009.

[32] W. Wang, S. Jin, X. Gao, K.-K. Wong, and M. R. McKay, "Power allocation strategies for distributed space-time codes in two-way relay networks," *IEEE Trans. Signal Process.*, vol. 58, no. 10, pp. 5331–5339, Oct. 2010.

[33] J. N. Laneman and G. W. Wornell, "Distributed space-time-coded protocols for exploiting cooperative diversity in wireless networks," *IEEE Trans. Inf. Theory*, vol. 49, no. 10, pp. 2415–2425, Oct. 2003.
[34] Y. Chang and Y. Hua, "Diversity analysis of orthogonal space-time modulation for distributed wireless relays," in *Proc. Int. Conf. Acoust., Speech, Signal Process.*, vol. 4. Montreal, QC, Canada, May 2004, pp. iv-561–iv-564.
[35] S. Yang and J.-C. Belfiore, "Optimal space–time codes for the MIMO amplify-and-forward cooperative channel," *IEEE Trans. Inf. Theory*, vol. 53, no. 2, pp. 647–663, Feb. 2007.
[36] D. To, J. Choi, and I.-M. Kim, "Error probability analysis of bidirectional relay systems using Alamouti scheme," *IEEE Commun. Lett.*, vol. 14, no. 8, pp. 758–760, Aug. 2010.
[37] M. K. Arti and M. R. Bhatnagar, "Performance analysis of two-way AF MIMO relaying of OSTBCs with imperfect channel gains," *IEEE Trans. Veh. Technol.*, vol. 63, no. 8, pp. 4118–4124, Oct. 2014.
[38] J. Yang, P. Fan, T. Q. Duong, and X. Lei, "Exact performance of two-way AF relaying in Nakagami-m fading environment," *IEEE Trans. Wireless Commun.*, vol. 10, no. 3, pp. 980–987, Mar. 2011.
[39] B. Maham, A. Hjørungnes, and B. S. Rajan, "Quasi-orthogonal design and performance analysis of amplify-and-forward relay networks with multiple-antennas," in *Proc. IEEE Wireless Commun. Netw. Conf. (WCNC)*, Sydney, NSW, Australia, Apr. 2010, pp. 1–6.
[40] M. Z. A. Khan, B. S. Rajan, and M. H. Lee, "On single-symbol and double-symbol decodable STBCs," in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, Berlin, Germany, Jun./Jul. 2003, p. 127.
[41] C. Yuen, Y. L. Guan, and T. T. Tjhung, "Full-rate full-diversity STBC with constellation rotation," in *Proc. IEEE VTC*, vol. 1. Apr. 2003, pp. 296–300.
[42] C. Yuen, Y. L. Guan, and T. T. Tjhung, "A class of four-group quasi-orthogonal STBC achieving full rate and full diversity for any number of antennas," in *Proc. IEEE 16th Int. Symp. Pers., Indoor Mobile Radio Commun.*, vol. 1. Yokohama, Japan, Sep. 2005, pp. 92–96.
[43] S. Karmakar and B. S. Rajan, "High-rate, multisymbol-decodable STBCs from Clifford algebras," *IEEE Trans. Inf. Theory*, vol. 55, no. 6, pp. 2682–2695, Jun. 2009.
[44] G. D. Forney and G. Ungerboeck, "Modulation and coding for linear Gaussian channels," *IEEE Trans. Inf. Theory*, vol. 44, no. 6, pp. 2384–2415, Oct. 1998.
[45] O. Tirkkonen, A. Boariu, and A. Hottinen, "Minimal non-orthogonality rate 1 space-time block code for 3+ Tx antennas," in *Proc. IEEE 6th Int. Symp. Spread-Spectr. Techn. Appl.*, Parsippany, NJ, USA, Sep. 2000, pp. 429–432.
[46] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge, MA, USA: Cambridge Univ. Press, 1985.
[47] K. J. R. Liu, A. K. Sadek, W. Su, and A. Kwasinski, *Cooperative Communications and Networking*. Cambridge, U.K.: Cambridge Univ. Press, 2009.
[48] W. Su and X.-G. Xia, "Signal constellations for quasi-orthogonal space-time block codes with full diversity," *IEEE Trans. Inf. Theory*, vol. 50, no. 10, pp. 2331–2347, Oct. 2004.
[49] J.-K. Zhang, J. Liu, and K. M. Wong, "Trace-orthonormal full-diversity cyclotomic space–time codes," *IEEE Trans. Signal Process.*, vol. 55, no. 2, pp. 618–630, Feb. 2007.

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